Universität der Bundeswehr München

Computable Functional Analysis and Probabilistic Computability

Dissertation
zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften (Dr. rer. nat.)

vorgelegt von
Dipl.-Math. Volker Bosserhoff
am 2. Oktober 2008

Tag der mündlichen Prüfung:	18. Dezember 2008
Vorsitzender der Kommission:	Prof. Dr. Markus Siegle
1. Berichterstatter:	Prof. Dr. Peter Hertling
2. Berichterstatter:	Prof. Dr. Vasco Brattka
3. Berichterstatter:	Prof. Dr. Cornelius Greither
4. Berichterstatter:	Prof. DDr. Stefan Schäffler

Universität der Bundeswehr München
Fakultät für Informatik

实事求是

邓小平

„Be practical and realistic.“

Deng Xiaoping

Contents

Chapter 1

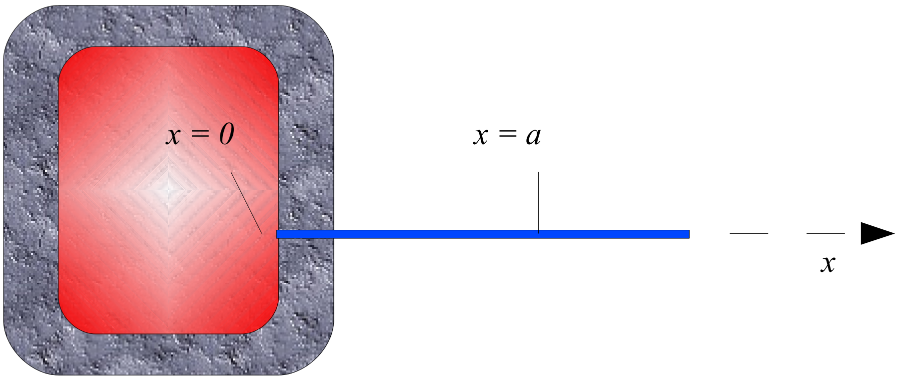
Introduction & Overview

Most of the content of the present work is motivated by the desire of gaining a better understanding of several results dealing with compact operators and their ill-posed inversion problems; those results were obtained by researchers from different schools of computational mathematics. In this chapter we will introduce and compare the different schools, survey the relevant results from the literature, and give an outline of the new results to be proved in this thesis.

Compact mappings and their inverses – a practical example

We shall begin with an example taken from [Gro93][1]: Suppose that we would like to control the temperature on the inner side of the pressure vessel of a nuclear reactor. A secure way to measure this temperature may be to have a metallic bar installed which passes through the wall of the vessel as shown in the figure. We imagine that the bar is very long so that we can model

[1]The books [Gro93] and [EHN96] contain many more examples from science and engineering which lead to mathematical problems of the same nature.



it as being infinitely long. Denote by $u(x,t)$ the temperature in the bar at point x at time t. We are interested in computing the function $f(t) := u(0,t)$ from observations of the function $g(t) := u(a,t)$. We assume that the initial temperature of the bar is 0, i.e.

$$u(x,0) = 0, \qquad x > 0,$$

and that the heat propagation in the bar fulfills the *one-dimensional heat equation*, i.e.

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \qquad 0 < x,t < \infty.$$

We furthermore assume that u is uniformly bounded. One can show (using a Laplace transform technique) that

$$g(t) = \frac{a}{2\sqrt{\pi}} \int \underbrace{\chi_{[0,t]}(\tau)\frac{\exp(-a^2/4(t-\tau))}{(t-\tau)^{3/2}}}_{=:k(t,\tau)} f(\tau)\,\lambda(d\tau) =: K(f)(t). \tag{1.1}$$

(Here and in the following, χ_M shall denote the characteristic function of the set M and λ shall denote Lebesgue measure.) (1.1) has the form of a *Fredholm integral equation of the first kind*. One has $k \in L_2(\lambda^2)$ which implies that K as given by (1.1) is a compact linear mapping from $L_2(\lambda)$ into $L_2(\lambda)$ (see e.g. [Gro80, p. 140]). We recall that a linear operator $F : X \to Y$ of Banach spaces X, Y is **compact** if the image $F(B_X)$ of the closed unit ball $B_X \subseteq X$ is relatively compact in Y. We are hence confronted with the problem of inverting the compact operator K.

Compact operators are a classical object of study in functional analysis (see e.g. [Meg98]). They can be thought of as having a strong *smoothing* effect on their operands. Unfortunately, the inverses of those smooth operators exhibit a particularly unpleasant behaviour: If X and Y are infinite dimensional and a compact operator $F : X \to Y$ is injective, then its inverse $F^{-1} :\subseteq Y \to X$ is necessarily discontinuous. The problem of computing an unbounded linear mapping is considered **ill-posed** (a term going back to Hadamard) because numerical methods (which are always approximative in nature) are bound to fail on them in general.

The present work is motivated by questions that arise when the computational properties of compact operators and their inverses are studied using different formal approaches to computing on continuous data.

Different models for scientific computation

Real numbers, differentiable functions, probability measures etc. are abstractions that are commonplace in scientific modelling. But a fundamental problem arises when it comes to implementing these abstractions in concrete computer programs: The computing machines that we use today are based on performing discrete steps in each of which a finite number of digits is manipulated. How shall such machines handle objects such as real numbers the complete description of which potentially requires infinitely many digits?

Today's computer hardware supports so called **floating-point** data types along with instructions for **floating-point arithmetic** [IEE85]. The floating-point numbers are a fixed finite set of rational numbers; floating-point arithmetic is an approximation to the actual arithmetic on these numbers and includes built-in rounding. The floating-point instructions also include boolean-valued operations like "equals" or "greater than". The time and the memory that are taken up by each arithmetic or boolean floating-point operation are bounded by constants; for complexity considerations, it hence seems reasonable to consider them as atomic. This is very close to the way that real numbers are thought of in classical[2] mathematics: the numbers themselves, as well as algebraic operations and comparisons are atomic primitives. The monograph [BCSS98] contains the foundations of a computability and complexity theory based on a class of abstract machines that store real numbers in single cells of their tapes and have comparisons as well as a finite number of rational functions as built-in atomic operations. This model of computation on real numbers became known as the **BSS-model** (because it was first defined in [BSS89]). The BSS-model is intended as a theoretical foundation for numerical analysis. It is an example of an **algebraic** or **real-number model** of computation.[3]

When real-number machines are employed to model floating-point computations, all issues that arise from the floating-point arithmetic's limited precision are neglected. So the numerical stability of an algorithm the correctness and complexity of which have only been proved in the real-number model must be studied in a second step. Usually the algorithm's floating-point implementation will only produce reasonable output on a restricted set of problem instances. For some applications, however, the floating-point approach and thus the real-number model seem to be fundamentally inadequate: A source of examples is the whole area of **computational geometry**, where naive algorithms are notorious for breaking down or silently producing qualitatively incorrect output due to numerical errors. (See [KMP$^+$04] for examples.)

When fixed-precision arithmetic is not sufficient for solving a certain problem, one will seek for a program that asks for the desired output precision and then automatically determines the degree of precision the input has to be be provided in[4] and the degree of precision the algebraic operations must be evaluated with in order to achieve the prescribed error bounds. This mode of computation is not modelled adequately by real-number machines. For example, the equality of two real numbers cannot be decided based on approximations of the numbers; a model that takes this into account must not have equality as a built-in predicate.

There is further critique of the real-number model concerning its suitability as a foundation for a practically relevant computability and complexity theory of subsets of Euclidean space (see

[2]The term "classical mathematics" is used here in contrast to "constructive mathematics" or "computable mathematics".

[3]Models of this type had been in use long before the introduction of the BSS-machine. See [BCSS98] for a historical survey.

[4]This precision may depend on the input itself, so the program may ask several times for better and better approximations.

[Bra03b, Par05, BC06]).

But what is a suitable model for studying which problems involving real numbers can be solved by computers and which cannot? The almost universally accepted model for digital computers is the **Turing machine** [Tur36, Tur37]. It has served as the foundation for successful theories of the computability [Rog87] and complexity [DK00] of discrete problems. Turing, however, had originally defined his machine for the purpose of computing on real numbers: The numbers should be represented by infinite strings of symbols on the input and output tapes. This already yields the modern definition of a **computable real number**: $x \in \mathbb{R}$ is computable if there is a Turing machine without an input tape that writes a binary expansion of x onto its output tape. The modern definition of a **computable real function** was later given by Grzegorczyk [Grz57] and Lacombe [Lac55]. Their work can be seen as the initiation of what is now called **computable analysis**. The present work is a contribution to that branch of theoretical computer science.

Computable analysis is very close to **constructive mathematics** [BB85, BV06]: In both disciplines, real numbers are not treated as atomic objects, but as sequences of discrete objects where each can be described finitely and delivers some more information on the real.

There are several mutually consistent approaches to computable analysis, e.g. the **axiomatic approach** [PER89] or the **oracle machine approach** [Ko91].[5] We shall concentrate on the **representation-based** approach, aka **type-two theory of effectivity (TTE)** [Wei00, BHW08]. TTE allows a unified treatment of uniform computability not only on real numbers and functions, but also on general metric and normed spaces, on open/closed/compact sets, on measures and random variables etc. The computational complexity of real numbers, real functions, compact subsets of $\mathbb{R}^n$ and a few other classes of objects has also been defined in a reasonable way based on representations [Wei00, Wei03].

Chapter 2 will be a compact introduction to all notions and facts from TTE that will be used in this thesis. We also include proofs of a number of useful propositions which we have not found in the literature. The standard reference for representation-based computable analysis is Weihrauch's monograph [Wei00].

Compact operators in computable functional analysis

The mathematical treatment of problems from science and engineering – in particular integral and differential equations – often requires the apparatus of functional analysis and involves objects less elementary than real numbers or continuous functions (e.g. L_p-spaces, Sobolev spaces, generalized functions, etc.). The treatment of such objects within computable analysis – we speak of **computable functional analysis** – was pioneered by Pour-El and Richards [PER89] and their collaborators. However, computable functional analysis can also be developed based

[5] See [Wei00, Chapter 9] for comparisons.

on representations. Many of the results obtained by the Pour-El and Richards school have by now been reproved and extended within TTE;[6] some topics were treated using TTE from the start.

We present a selection of relevant references: Brattka compared several representations of computable linear operators on Banach spaces in [Bra03a]. He investigated the computable content of classical Banach space principles in [Bra08a, Bra08b, Bra]. The computability of spectra of self-adjoint linear operators on Hilbert spaces was studied by Brattka and Dillhage [BD05, Dil08]. Brattka and Yoshikawa [BY06] is a source for computable versions of many results on Hilbert spaces. A computability theory of generalized functions was developed by Zhong and Weihrauch in [ZW03]. The same authors studied the computability of the solution operators of several classes of PDEs in [WZ02, ZW03, WZ05, WZ06a, WZ06b, WZ07].

The treatment of compact operators in computable functional analysis was initiated by Brattka and Dillhage in [BD07]. Our work is tied in with that publication. Brattka and Dillhage defined a representation of the space of compact operators on computable Banach spaces (see Chapter 2) and proved effective versions of a number of classical theorems on compact operators. However, they made the additional assumption that the Banach spaces under consideration possess computable **Schauder bases**. We will review these notions as well as Brattka and Dillhage's results in Subsection 3.1. It is well-known that computable *Hilbert* spaces always possess computable Schauder bases. In order to gain a better understanding of computable bases in the general Banach space setting, we asked the following question: *Given a computable Banach space that possesses a basis. Does it possess a computable basis?* Chapter 3 is devoted to the construction of a counterexample. Our construction builds on deep results from classical functional analysis.

The degree of uncomputability caused by ill-posed problems

At the beginning of this chapter we already saw how the problem of inverting a compact operator arises in an application. Such inverse problems typically cause great computational difficulties as solving would require the evaluation of discontinuous operators. Another familiar example of this phenomenon is differentiation considered as the inversion of integration: It is well-known (see [Wei00]) that the integral of a continuous function on $[0, 1]$ can be computed while the derivative of a computable differentiable function may be uncomputable. An inversion problem may, however, become solvable if additional information on the source object is available: The derivative of a computable differentiable function can be computed if the derivative itself has a bounded derivative and an upper bound for the absolute value of this second derivative is provided as additional input. A general theory of such **source conditions** and their exploitation

[6] The TTE versions usually make stronger statements than the original ones. E.g. Pour-El and Richards [PER89] proved that the spectrum of a computable self-adjoint operator is a computably enumerable compact set; Brattka and Dillhage [BD05] proved that a name of the operator can be computably transformed into a name of the spectrum.

for the numerical solution of inverse problems was initiated by Tikhonov and Phillips and is known as **regularization** (see [Gro77, EHN96]).

In Chapter 4 we will first recall the definition of the **generalized inverse** $T^\dagger$ of a bounded linear mapping $T : X \to Y$ of Hilbert spaces X, Y. We then use a computable reduction of functions, invented by Weihrauch [Wei92], to characterize how uncomputable the evaluation map $(T, y) \mapsto T^\dagger y$ is. One direction of the reduction is based on Tikhonov regularization. The other direction uses Brattka's version [Bra99] of Pour-El and Richards' [PER89] **First Main Theorem**.

Ill-posed problems in information-based complexity

A school of numerical mathematics that uses a real-number model – originally in a somewhat informal way, later formalized in [Nov95] and [Pla96, Section 2.9] – is **information-based complexity (IBC)** [TWW88, TW98]. IBC is centered around the idea that the cost of computing a numerical operator depends on the available information on the input. Here, the input is typically an element of the unit ball of a normed linear function space and information is retrieved by applying elements of a prescribed set of real valued functionals (such as evaluation functionals or Fourier transforms) to the input.

A computational problem in IBC is made up of a set of problem elements D, a solution operator $S : D \to Y$, where Y is a normed space, and a set Λ of admissible information functionals. The problem is considered **solvable in the worst-case setting** if for every prescribed error bound $\varepsilon > 0$, there is a real-number machine with functional oracles from Λ that computes a mapping $\Psi_\varepsilon : D \to Y$ with

$$\sup_{x \in D} \|S(x) - \Psi_\varepsilon(x)\| \leq \varepsilon. \tag{1.2}$$

If one replaces condition (1.2) by

$$\int_D \|S(x) - \Psi_\varepsilon(x)\|^2 \, \nu(dx) \leq \varepsilon$$

or

$$\nu(\{x \in D \;:\; \|S(x) - \Psi_\varepsilon(x)\| > \varepsilon\}) \leq \varepsilon$$

for a fixed probability measure ν on D, then the problem is considered **solvable in the average-case setting** or **solvable in the probabilistic setting**, respectively, for ν. The **cost** of computing $\Psi_\varepsilon(x)$ is the sum of the **information cost** (i.e. the number of functional oracles invoked) and the **combinatorial cost** (i.e. the number of algebraic operations and branches performed). The combinatorial cost typically turns out to be proportional to the information cost, so the focus of attention in IBC is on information cost. Furthermore, many algorithms found in IBC are rather simple on the combinatorial side; consider for example the so called **linear algorithms** that are optimal for a large class of problems (see [TWW88, Section 4.5.5]).

Considering the definition of *solvability* just presented, one observes that it does not demand uniformness in the error level, i.e. one does not ask for one single machine which takes the error bound as an additional input and then computes the solution operator up to this given precision. In fact, uniform algorithms *can* be studied within the real-number model, but the possibility of using *magic constants* – which are real numbers that store an infinite amount of information which a real-world machine would have to compute itself – leads to somewhat unrealistic results [NW99]. In particular, the problem of choosing suitable information functionals for any given error level is obscured by the presence of magic constants: Names of nearly optimal functionals for every error level could be stored in a single magic constant. In reality, however, the computation of good information functionals on Turing machines may be a very complex task (see [Bos08b] for an example).

How does the computational intractability of linear ill-posed problems manifest itself in IBC? Werschulz [Wer87] proved that any algorithm (in the IBC sense) using continuous linear information functionals has an infinite worst-case error when applied to approximate an unbounded linear operator, i.e. linear ill-posed problems are not solvable in IBC's worst-case setting. In the same paper, however, Werschulz proved a positive result for the average-case setting for **Gaussian measures**; this result was later generalized by Werschulz and others (see Chapter 7 for more references).

Werschulz' results are surveyed by Traub and himself in [TW94] and [TW98, Chapter 6]. In the latter reference, the authors draw an analogy between Werschulz' negative result on the one side and Pour-El and Richards' First Main Theorem on the other side. We have already mentioned the First Main Theorem above. It implies that unbounded linear operators are uncomputable and typically even map some computable points to uncomputable points.[7] (The precise statement is given in Chapter 2.)

As a transition from the worst-case to the average-case setting makes ill-posed problems solvable in the sense of IBC, Traub and Werschulz ask whether such a transition is also possible in computable analysis [TW98, p. 60]:

> Is every (measurable) linear operator computable on the average for Gaussian measures?

Traub and Werschulz do not tell what they mean by "computable on the average". This was the starting point for our fundamental study of several notions of probabilistic computability and the computability of Gaussian measures.

[7] An analogy between Werschulz' negative result and the First Main Theorem is also drawn in [Tra99], where the relative length of Pour-El and Richards' proof as compared to Werschulz' prove is taken as an argument for the superiority of the real-number model over the Turing machine model as a foundation for numerical analysis. We consider this comparison a little unfair: The mere *uncomputability* of unbounded operators follows directly from their discontinuity – this is a fundamental and easily provable fact. The hard part of the First Main Theorem is that some computable points are mapped to uncomputable points.

Probabilistic computability

Ko [Ko91] defines the notion of a **computably approximable** real function. This definition can be considered as computable analysis' analogue to IBC's probabilistic setting. Parker's [Par03, Par05, Par06] definition of **decidability up to measure zero** of a subset of $\mathbb{R}^n$ goes in a similar direction. An analogue to IBC's average-case setting, however, has not yet been considered in computable analysis.

In Chapter 5 we provide the foundations of a theory of probabilistic computability of mappings from general represented spaces into metric spaces. We extend Ko's and Parker's ideas to more general classes of mappings, but we also define the new notion of **computability in the mean** which corresponds to IBC's average-case setting. The latter definition builds on a suggestion by Hertling [Her05]. We shall also define representations that are tailor-made for mappings computable in the respective probabilistic sense. In the spirit of TTE's **Representation Theorem** [Wei00, Sch02c], we give characterizations of the mappings in the ranges of those representations. We furthermore study the mutual relations between the different computability concepts. Finally, we prove theorems on effective composition and vector-valued integration of probabilistically computable mappings.

Gaussian measures

The best-studied class of probability measures on infinite-dimensional spaces is the class of **Gaussian measures**. The best-known representative of this class is **Wiener measure**, i.e. the distribution of random **Brownian motion** (see e.g. [Kal02]). Most of the IBC results in the average-case setting were obtained with the assumption that the underlying measure is Gaussian.

In Chapter 6, we first collect a number of useful properties of Gaussian measures. We use the modern literature (in particular [Bog98]) to gain understanding of the structure of linear Gaussian random elements; we will have Werschulz' result on the average-case solvability of linear ill-posed problems as a corollary. We then look at Gaussian measures from the point of view of computable analysis. We define two representations of Gaussian measures on separable Hilbert spaces – one with a more "algebraic", the other with a more "topological flavour". We prove these representations to be computably equivalent. This result can be seen as an effective version of what is sometimes called the **Mourier Theorem**. The proof utilizes results from Chapter 5.

Probabilistic computability of unbounded inverse operators

In Chapter 7 – the final part of this thesis – we apply the definitions and theorems from Chapters 4, 5, and 6 to interpret and answer Traub and Werschulz' question on the "average-case computability" of linear operators.

We sketch the results: If one does not ask for a computable approximator that is uniform in the operator and error level, Werschulz' result can be transferred easily into computability theory. But what if one demands more uniformness? We study this question for the special, yet practically very important case that the operator to be approximated is the (generalized) inverse of a bounded operator of Hilbert spaces: The bounded operator and its adjoint as well as the desired average error and the underlying Gaussian measure are provided as inputs. It turns out that this information is not sufficient to compute the inverse on the average. In fact, we can even give an example of a computable injective compact self-adjoint endomorphism of the sequence space ℓ_2 whose inverse does not fulfill a very weak interpretation of "computable on the average" for a certain Gaussian measure of very simple structure. Like in Chapter 4, we use a computable reducibility of functions to characterize the corresponding degree of uncomputability. A positive result, however, is possible if one assumes that additional information on the L_2-norm of the inverse is available; with this number as additional input, one can even employ a rather strong interpretation of "computable on the average".

The findings of Chapter 7 can be considered as the main results of this thesis. They demonstrate how the Turing machine model allows a more detailed exploration of the limits of what is computationally possible than the real-number model.

Acknowledgment

First of all I would like to thank my supervisor Peter Hertling for accepting me as a student, helping me find an interesting subject, organizing financial funding for my work, and supporting me throughout the entire process of the preparation of this thesis. I am especially grateful for the time he spent reading drafts of my papers and for the valuable comments and suggestions he made.

I thank Vasco Brattka for his hospitality and the fruitful discussions during my visit to Cape Town.

My thanks also go to all members of the *Institut für Theoretische Informatik und Mathematik* at the University of the Armed Forces in Munich for providing a very comfortable environment over the past three years.

The entire project would have never been possible without the constant support by my family and friends. In particular, I thank Heike Bosserhoff for proof-reading parts of the thesis.

The *Deutsche Forschungsgemeinschaft* (DFG) funded my work in 2006 and the first half of 2007 (grant HE 2489/4-1). I additionally received a grant from the DFG and the *National Science Foundation of China* to visit the University of Nanjing in autumn 2006 and a grant from the DFG and the *South African National Research Foundation* to visit the University of Cape Town in spring 2008.

The present version of this thesis already incorporates a number of minor changes and corrections suggested by the referees. Some of the material was published in the articles [Bos07, Bos08c, Bos08a, Bos08d] and presented at the *Computability and Complexity in Analysis* conferences in 2006, 2007, and 2008.

Notational conventions

The natural numbers

The natural numbers $\mathbb{N}$ are understood to include zero. We always consider the discrete topology on $\mathbb{N}$.

Intervals, norms, inner products

We write intervals with a semicolon. So the open interval $(a; b)$ is distinguished from the pair (a, b).

We will almost never distinguish the norms of different normed spaces notationally, but write $\|\cdot\|$ for all of them. Which norm is meant will always be clear from what is the argument.

Inner products of Hilbert spaces are always written $\langle \cdot \mid \cdot \rangle$.

Characteristic functions

If M is a set and A is a subset of M, then $\chi_A : M \to \{0, 1\}$ with

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

is the **characteristic function** of A.

Finite and infinite sequences

If M is a set, put

$$M^{<\omega} := \bigcup_{n \in \mathbb{N}} M^n.$$

Here $M^0 = ()$ is the **empty sequence**. Elements of $M^{<\omega}$ are sometimes called **finite sequences**. Let M^ω be the set of (infinite) sequences whose terms are in M. We identify this set with the set of mappings from $\mathbb{N}$ to M.

The **range** of a sequence $(x_i)_i \in M^\omega$ is the set

$$\mathrm{range}((x_i)_i) := \{x \in M \ : \ (\exists\, i \in \mathbb{N})\ (x = x_i)\}.$$

For a finite sequence $(x_1, \ldots, x_n) \in M^{<\omega}$, put

$$\mathrm{range}((x_1, \ldots, x_n)) := \{x \in M \ : \ (\exists\, 1 \leq i \leq n)\ (x = x_i)\}.$$

If $(v_1, \ldots, v_n), (w_1, \ldots, w_m) \in M^{<\omega}$ and $(p_i)_i \in M^\omega$, we define concatenations in the usual fashion:

$$\begin{aligned}(v_1, \ldots, v_n)(w_1, \ldots, w_m) &:= (v_1, \ldots, v_n, w_1, \ldots, w_m),\\ (v_1, \ldots, v_n)(p_i)_i &:= (v_1, \ldots, v_n, p_0, p_1, \ldots).\end{aligned}$$

If $v \in M^{<\omega}$, $W \subseteq M^{<\omega}$, and P is a subset of $M^{<\omega}$ or of M^ω, define

$$\begin{aligned}vP &:= \{vp \ : \ p \in P\},\\ WP &:= \bigcup_{w \in W} wP.\end{aligned}$$

If M is endowed with a topology τ, we will automatically endow M^n ($n \in \mathbb{N}$) and M^ω with the product topologies $\tau^n := \prod_{i=1}^n \tau$ and $\tau^\omega := \prod_{i \in \mathbb{N}} \tau$, respectively. This in particular means that $\mathbb{N}^\omega$ is the **Baire space** (see e.g. [Wei00, Exercise 2.2.9]).

Tupling

For every non-empty word $\mathbf{w} = w_1 \ldots w_n$ over the alphabet $\{*, \omega\}$, there is a continuous bijective **tupling function**

$$\langle \cdot, \ldots, \cdot \rangle_{\mathbf{w}} : A_{w_1} \times \cdots \times A_{w_n} \to B_{\mathbf{w}},$$

where

$$A_v := \begin{cases} \mathbb{N}^{<\omega} & \text{if } v = *,\\ \mathbb{N}^\omega & \text{if } v = \omega \end{cases}$$

and

$$B_{\mathbf{w}} := \begin{cases} \mathbb{N}^{<\omega} & \text{if } w_1 = \ldots = w_n = *,\\ \mathbb{N}^\omega & \text{otherwise.} \end{cases}$$

Such tupling functions can be defined in a canonical way. We in particular assume that the tuplings and their inverses are computable by type-two Turing machines, which we will introduce below.

Partial and multi-valued mappings

Let M, N be sets. A **partial multi-valued mapping** f is a correspondence $f = (\Phi, M, N)$, i.e. $\Phi \subseteq M \times N$. We write $f :\subseteq M \rightrightarrows N$. Φ is called the **graph** of f and will be denoted by $\mathrm{graph}(f)$. The **domain** of f is defined by

$$\mathrm{dom}(f) := \{x \in M \ : \ (\exists\, y \in N)\, ((x, y) \in \Phi)\}.$$

If A is a subset of M we write

$$f(A) := \{y \in N \ : \ (\exists\, x \in A)\, ((x, y) \in \Phi)\}.$$

The **range** of f is then defined by $\mathrm{range}(f) := f(M)$. For single-valued $A = \{x\}$, we also write $f\{x\}$ or $f(x)$ instead of $f(\{x\})$; we will, however, only do so if $x \in \mathrm{dom}(f)$. f is called **surjective** if $\mathrm{range}(f) = N$. The **inverse** f^{-1} of f is the partial multi-valued mapping (Φ^{-1}, N, M), where

$$\Phi^{-1} := \{(y, x) \in N \times M \ : \ (x, y) \in \Phi\}.$$

If A is a subset of M, let $f|_A$ be the partial multi-valued mapping $(\Phi|_A, A, N)$, where

$$\Phi|_A := \{(x, y) \in \Phi \ : \ x \in A\}.$$

If A is a subset of N, let $f|^A$ be the partial multi-valued mapping $(\Phi|^A, M, A)$, where

$$\Phi|^A := \{(x, y) \in \Phi \ : \ y \in A\}.$$

If order to define a partial multi-valued mapping $f :\subseteq M \rightrightarrows N$, we shall sometimes use the convenient notation

$$f(x) \ni y \quad :\Longleftrightarrow \quad P(x, y), \tag{1.3}$$

where P is a predicate defined on a subset of $M \times N$. (1.3) should be read as

$$\mathrm{graph}(f) := \{(x, y) \in M \times N \ : \ P \text{ is defined and true on } (x, y)\}.$$

$f :\subseteq M \rightrightarrows N$ is called **total** if $\mathrm{dom}(f) = M$. We write $f : M \rightrightarrows N$ to indicate that f is total.

$f :\subseteq M \rightrightarrows N$ is called **single-valued** if $f(x)$ has cardinality one for every $x \in \mathrm{dom}(f)$. We write $f :\subseteq M \to N$ to indicate that f is single-valued and that (notwithstanding the definition above) we understand by $f(x)$ the unique $y \in N$ with $(x, y) \in \mathrm{graph}(f)$ (instead of the one-element set $\{y\}$).

If $f :\subseteq M \rightrightarrows N$ is total and single-valued, then f is simply a mapping in the usual sense and we write $f : M \to N$ to indicate this.

If M and N are endowed with topologies, then $f :\subseteq M \to N$ is called **continuous** if $f^{-1}(U)$ is relatively open in $\mathrm{dom}(f)$ for every open $U \subseteq N$.

Chapter 2

Representation-Based Computable Analysis

2.1 Computability and naming systems

2.1.1 Computability on $\mathbb{N}$ and $\mathbb{N}^\omega$

For the exact definition of **type-two Turing machines** and the partial functions computed by them, we refer to [Wei00]. Let us only mention a few aspects: A type-two machine M works much like an ordinary Turing machine. It has three one-way infinite tapes whose cells are either blank or contain a symbol from the alphabet $\Sigma := \{\mathtt{0}, \mathtt{1}\}$: The input tape is read-only; the work tape is random-access; the output tape is write-only and the head may only move to the right. M has a finite control just like an ordinary Turing machine. The definition of M is amended by a **type specification** $(A_1, A_2) \in \{\mathbb{N}, \mathbb{N}^\omega\}^2$; A_1 is the set from which input is expected, A_2 is the set that the output is supposed to be in. Elements of $\mathbb{N}$ or $\mathbb{N}^\omega$ are written onto the input and output tapes by encoding them as elements of $\Sigma^{<\omega}$ or Σ^ω, respectively, in a canonical (bi-continuous) way. M computes a function $f_\mathrm{M} :\subseteq A_1 \to A_2$ which is defined as follows: If $A_2 = \mathbb{N}$ then f_M is defined on all $p \in A_1$ on which the computation of M enters a halting state; $f_M(p)$ is defined as the number whose codeword has been written onto the output type by then. If $A_2 = \mathbb{N}^\omega$ then f_M is defined on all $p \in A_1$ on which the computation of M produces infinitely many output symbols that form a valid codeword for an element of $\mathbb{N}^\omega$; $f_\mathrm{M}(p)$ is then defined to be this element. A single element $p \in \mathbb{N}^\omega$ is **computable** if there is a type-two machine that produces output p when run with an empty input tape.

Let $A_1, A_2 \in \{\mathbb{N}, \mathbb{N}^\omega\}$. A partial function $f :\subseteq A_1 \to A_2$ is called **computable** if there is a type-two Turing machine M such that $\mathrm{dom}(f) \subseteq \mathrm{dom}(f_\mathrm{M})$ and $f(p) = f_\mathrm{M}(p)$ for every $p \in \mathrm{dom}(f)$.

For the following fundamental result see [Wei00, Theorem 2.2.3]:

Theorem 2.1 (Main Theorem of TTE). *Let $A_1, A_2 \in \{\mathbb{N}, \mathbb{N}^\omega\}$. Every computable $f :\subseteq A_1 \to A_2$ is continuous.* □

2.1.2 Computability via naming systems

One uses *(multi-)naming systems*[1] to lift computability from $\mathbb{N}$ and $\mathbb{N}^\omega$ to other sets. If M is a set then a **multi-naming system** for M is a surjective partial multi-valued mapping $\delta :\subseteq A \rightrightarrows M$, where $A \in \{\mathbb{N}, \mathbb{N}^\omega\}$. In this case put $\mathrm{TYPE}(\delta) := A$. If $\mathrm{TYPE}(\delta) = \mathbb{N}$, then δ is in particular called a **multi-numbering**; if $\mathrm{TYPE}(\delta) = \mathbb{N}^\omega$, then δ is in particular called a **multi-representation**. A δ-**name** of $x \in M$ is an element p of $\mathrm{TYPE}(\delta)$ with $\delta(p) \ni x$. A point in M is δ-**computable** if it has a computable δ-name.

Let M_1, M_2 be sets with multi-naming systems δ_1, δ_2. Then $g :\subseteq \mathrm{TYPE}(\delta_1) \to \mathrm{TYPE}(\delta_2)$ is a (δ_1, δ_2)-**realization** of $f :\subseteq X_1 \rightrightarrows X_2$ if any δ_1-name p of any $x \in \mathrm{dom}(f)$ is in $\mathrm{dom}(g)$, and $g(p)$ is a δ_2-name of some element of $f(x)$. f is called (δ_1, δ_2)-**continuous** if it allows a continuous (δ_1, δ_2)-realization. Denote by $\Lambda(\delta_1 \rightrightarrows \delta_2)$ the set of all (δ_1, δ_2)-continuous partial multi-valued mappings; $\Lambda(\delta_1 \to \delta_2)$ shall be the subset of $\Lambda(\delta_1 \rightrightarrows \delta_2)$ that contains exactly the single-valued mappings; $\Lambda(\delta_1 \rightrightarrows \delta_2)_{\mathrm{TOT}}$ shall be the subset of $\Lambda(\delta_1 \rightrightarrows \delta_2)$ that contains exactly the total mappings; also of course

$$\Lambda(\delta_1 \to \delta_2)_{\mathrm{TOT}} := \Lambda(\delta_1 \to \delta_2) \cap \Lambda(\delta_1 \rightrightarrows \delta_2)_{\mathrm{TOT}}.$$

f is called (δ_1, δ_2)-**computable** if it allows a computable (δ_1, δ_2)-realization.

Let δ, δ' be multi-naming systems of sets M, M' with $M \subseteq M'$. We say that δ is **continuously reducible** to δ' if the embedding of M into M' is (δ, δ')-continuous; we write $\delta \le_t \delta'$. If $M = M'$, $\delta \le_t \delta'$, and $\delta' \le_t \delta$, then we say that δ and δ' are **continuously equivalent**; we write $\delta \equiv_t \delta'$. Requiring the embedding of M into M' to be even (δ, δ')-computable yields the notion of **computable reducibility** of δ to δ'; we write $\delta \le \delta'$. Accordingly, δ and δ' are **computably equivalent** if $\delta \le \delta'$ and $\delta' \le \delta$; we write $\delta \equiv \delta'$.

The majority of multi-naming systems considered in this text will be single-valued. In this case, we will simply call them **naming systems**; we will in particular speak of **numberings** and **representations**. If δ is a single-valued naming system, we shall also identify the singleton sets $\delta(p)$, $p \in \mathrm{dom}(\delta)$, with their single elements.

2.1.3 Some constructions with naming systems

Fix some $n \in \mathbb{N}$, $n \ge 1$. For $1 \le i \le n$ let M_i be a set with a multi-naming system $\delta_i :\subseteq A_i \rightrightarrows M_i$. Define the multi-naming system $[\delta_1, \ldots, \delta_n]$ of $M_1 \times \cdots \times M_n$ by

$$[\delta_1, \ldots, \delta_n]\langle r_1, \ldots, r_n \rangle_{\mathrm{w}} \ni (x_1, \ldots, x_n) \quad :\Longleftrightarrow \quad (\forall\, 1 \le i \le n)\ (\delta(r_i) \ni x_i).$$

[1]Multi-naming systems are not treated in [Wei00], but were defined first in [Sch02b].

Here $\mathbf{w} = w_1 \cdots w_n$ is given by

$$w_i = \begin{cases} * & \text{if } A_i = \mathbb{N}, \\ \omega & \text{if } A_i = \mathbb{N}^\omega \end{cases}.$$

For abbreviation, we put

$$[\delta]^n := [\underbrace{\delta, \dots, \delta}_{n\text{-times}}]$$

for every $n \geq 1$.

If δ is a multi-naming system of a set M, then a multi-naming system $[\delta]^{<\omega}$ of $M^{<\omega}$ is given by

$$[\delta]^{<\omega}\langle n, s\rangle_{*b} := \begin{cases} [\delta]^n(s) & \text{if } n > 0, \\ () & \text{else,} \end{cases}$$

where $b = *$ if δ is a multi-numbering, and $b = \omega$ if δ is a multi-representation.

Our next aim is to define a natural multi-representation for partial multi-valued mappings between sets with multi-naming systems. We will do so by modifying a definition by Weihrauch[2] (see [Wei00, Definition 3.3.13]).

Define

$$\begin{aligned} F^{**} &:= \{f \ : \ f :\subseteq \mathbb{N} \to \mathbb{N}\}, \\ F^{*\omega} &:= \{f \ : \ f :\subseteq \mathbb{N} \to \mathbb{N}^\omega\}, \\ F^{*\omega} &:= \{f \ : \ f :\subseteq \mathbb{N}^\omega \to \mathbb{N} \text{ is continuous}\}, \\ F^{\omega\omega} &:= \{f \ : \ f :\subseteq \mathbb{N}^\omega \to \mathbb{N}^\omega \text{ is continuous}\}. \end{aligned}$$

For $a, b \in \{*, \omega\}$, we construct a multi-representation η^{ab} of F^{ab}: Define

$$\alpha^*(n) := \{n\}$$

for all $n \in \mathbb{N}$, and

$$\begin{aligned} \alpha^\omega\langle 0, n\rangle_{**} &:= \mathbb{N}^\omega, \\ \alpha^\omega\langle k, \langle n_1, \dots, n_k\rangle_{*^k}\rangle_{**} &:= (n_1, \dots, n_k)\mathbb{N}^\omega \end{aligned} \tag{2.1}$$

for all $n, k, n_1, \dots, n_k \in \mathbb{N}$, $k \geq 1$. Define $p \in \mathbb{N}^\omega$ to be an η^{ab}-name of $f \in F^{ab}$ if, and only if,

(1.) for every $\langle r, s\rangle_{**} \in \operatorname{range}(p)$, one has

$$f(\alpha^a(r)) \subseteq \alpha^b(s).$$

(The information on f is correct.)

[2] Weihrauch defines a representation $[\delta_1 \to \delta_2]$ of the set of all (δ_1, δ_2)-continuous *total* mappings from a set M_1 with representation δ_1 into a set M_2 with representation δ_2. The advantage of the definition that we use shows, for example, in the formulation of item (1) of Lemma 2.20.

(2.) for every $(v, w) \in \mathrm{graph}(f)$ and every open neighborhood U of w there exists $\langle r, s\rangle_{**} \in \mathrm{range}(p)$ with
$$v \in \alpha^a(r) \quad \text{and} \quad w \in \alpha^b(s) \subseteq U.$$
(The information on f is complete.)

Lemma 2.2. *$f \in F^{ab}$ is computable if, and only if, f has a computable η^{ab}-name.* □

If M_1, M_2 are sets with multi-naming systems δ_1, δ_2, then a multi-representation $[\delta_1 \rightrightarrows \delta_2]$ of $\Lambda(\delta_1 \rightrightarrows \delta_2)$ is given by

$$[\delta_1 \rightrightarrows \delta_2](p) \ni f \quad :\Longleftrightarrow \quad \eta^{ab}(p) \text{ contains a } (\delta_1, \delta_2)\text{-realization of } f,$$

where $a, b \in \{*, \omega\}$ are chosen according to the types of δ_1, δ_2. Define

$$\begin{aligned}
[\delta_1 \to \delta_2] &:= [\delta_1 \rightrightarrows \delta_2]|^{\Lambda(\delta_1 \to \delta_2)} \\
[\delta_1 \rightrightarrows \delta_2]_{\mathrm{TOT}} &:= [\delta_1 \rightrightarrows \delta_2]|^{\Lambda(\delta_1 \rightrightarrows \delta_2)_{\mathrm{TOT}}} \\
[\delta_1 \to \delta_2]_{\mathrm{TOT}} &:= [\delta_1 \rightrightarrows \delta_2]|^{\Lambda(\delta_1 \to \delta_2)_{\mathrm{TOT}}}.
\end{aligned}$$

Note that $[\delta_1 \to \delta_2]_{\mathrm{TOT}}$ is single-valued if δ_2 is single-valued.

We collect a number of properties of $[\delta_1 \rightrightarrows \delta_2]$. The proofs are similar to the proofs of the corresponding results in [Wei00].

Lemma 2.3 (Properties of the function space representation)**.** *Let M_1, M_2 be sets with multi-naming systems δ_1, δ_2. Let the evaluation map*

$$\mathrm{eval} :\subseteq \Lambda(\delta_1 \rightrightarrows \delta_2) \times M_1 \rightrightarrows M_2$$

be defined by

$$\mathrm{dom}(\mathrm{eval}) := \{(f, x) \ : \ x \in \mathrm{dom}(f)\}$$

and

$$\mathrm{eval}(f, x) := f(x).$$

(1) eval *is $([[\delta_1 \rightrightarrows \delta_2], \delta_1], \delta_2)$-computable.*

(2) If δ is a representation of $\Lambda(\delta_1 \rightrightarrows \delta_2)$ such that eval *is $([\delta, \delta_1], \delta_2)$-computable, then*

$$\delta \leq [\delta_1 \rightrightarrows \delta_2].$$

(3) $f \in \Lambda(\delta_1 \rightrightarrows \delta_2)$ is (δ_1, δ_2)-computable if, and only if, f has a computable $[\delta_1 \rightrightarrows \delta_2]$-name.

(4) If δ_1', δ_2' are further multi-naming systems of M_1, M_2 with $\delta_1 \le \delta_1'$ and $\delta_2' \le \delta_2$, then

$$[\delta_1' \rightrightarrows \delta_2'] \le [\delta_1 \rightrightarrows \delta_2].$$

□

Lemma 2.4 (Type conversion)**.** *Let M_1, M_2, M_3 be sets with multi-naming systems $\delta_1, \delta_2, \delta_3$. A mapping*

$$G :\subseteq M_1 \to \Lambda(\delta_2 \rightrightarrows \delta_3)$$

is $(\delta_1, [\delta_2 \rightrightarrows \delta_3])$-computable if the mapping

$$F :\subseteq M_1 \times M_2 \rightrightarrows M_3$$

with

$$\mathrm{graph}(F) := \{((x, y), z) \ : \ x \in \mathrm{dom}(G),\ (y, z) \in \mathrm{graph}(G(x))\}$$

is $([\delta_1, \delta_2], \delta_3)$-computable. □

Type conversion will often be used implicitly in proofs: In order to show that a name of a mapping can be computed relative to available information, we only need to show how to compute the mapping pointwise.

We introduce our first concrete naming system: Let the total numbering $\nu_{\mathbb{N}}$ of $\mathbb{N}$ be simply the identity on $\mathbb{N}$. Given a multi-naming system δ of a set M, we can now easily construct a multi-naming system $[\delta]^\omega$ of M^ω by putting

$$[\delta]^\omega := [\nu_{\mathbb{N}} \to \delta]_{\mathrm{TOT}}.$$

2.1.4 Decidability and enumerability via multi-naming systems

We introduce the **Sierpiński representation** $\int$ of $\{0, 1\}$:

$$\int(r) = \begin{cases} 1 & \text{if } 1 \in \mathrm{range}(r), \\ 0 & \text{otherwise.} \end{cases}$$

Let M be a set with a multi-naming system δ. A representation $[\delta]^{\mathrm{en}}$ of the set of all subsets of M whose characteristic functions are $(\delta, \int)$-continuous is given by

$$[\delta]^{\mathrm{en}}(r) = A \quad :\Longleftrightarrow \quad [\delta \to \int]_{\mathrm{TOT}}(r) = \chi_A.$$

A subset A of M is δ-**computably enumerable (c.e.)** if it is $[\delta]^{\mathrm{en}}$-computable. A is called δ-**decidable** if both A and $M \setminus A$ are δ-c.e.

2.2 Computing on topological spaces

We will now consider topological spaces with representations.

2.2.1 Spaces with continuous representations are Lindelöf

We note a useful property that a topological space must necessarily fulfill in order to allow a continuous representation. Recall that a topological space is **Lindelöf** if every open cover of the space contains a countable subcover. A space is **hereditarily Lindelöf** if every topological subspace is Lindelöf. Also recall that a topological space is **second-countable** if its topology has a countable basis.

Lemma 2.5 (Continuously represented spaces are Lindelöf)**.** *Let M be a topological space and let δ be a continuous naming system for M. Then M is hereditarily Lindelöf.*

Proof. If δ is a numbering, then M is necessarily countable and hence trivially hereditarily Lindelöf. So suppose that δ is a representation. It is enough to show that M is Lindelöf; the hereditarity then follows because if A is a subset of M then $\delta|^A$ is a continuous representation of A. As continuous mappings map Lindelöf spaces to Lindelöf spaces (see [Wil70, Theorem 16.6.a]), it is enough to show that $\mathrm{dom}(f)$ is Lindelöf. Being a subspace of the second-countable Baire space, $\mathrm{dom}(f)$ is itself second-countable (see [Wil70, Theorem 16.2.b]). Every second-countable space is Lindelöf (see [Wil70, Theorem 16.9.a]). $\square$

2.2.2 Admissibility

A representation of a topological space X can be better adjusted to the space's topology than just by being continuous: A representation δ of X is **admissible** with respect to the topology of X if δ is continuous and every continuous representation of X is continuously reducible to δ. This definition of admissibility is Schröder's [Sch02a, Sch02c] generalization of a definition by Kreitz and Weihrauch [KW85, Wei00]. It is known (see [Sch02c, Theorem 13]) that a topological space allows an admissible representation if its topology is T_0 and has a countable pseudobase.[3] One of the most important properties of admissible representations is reflected in the following theorem (see [Sch02c, Theorem 4]). Recall that a mapping $f : X_1 \to X_2$ of topological spaces X_1, X_2 is **sequentially continuous** if $(f(x_n))_n$ converges to $f(x_\infty)$ for every $x_\infty \in X_1$ and every sequence $(x_n)_n \in X_1^\omega$ that converges to x_∞.

[3] A family $\beta \subseteq 2^X$ is a **pseudobase** of a topological space X if for every open $U \subseteq X$, every $x_\infty \in U$, and every sequence $(x_n)_n \in X^\omega$ that converges to x_∞, there is a $B \in \beta$ and an $n_0 \in \mathbb{N}$ such that $\{x_\infty\} \cup \{x_n : n \geq n_0\} \subseteq B \subseteq U$.

Theorem 2.6 (Kreitz-Weihrauch-Schröder Representation Theorem)**.** *Let X_1, X_2 be topological spaces with admissible representations δ_1, δ_2. A partial mapping $f :\subseteq X_1 \to X_2$ is sequentially continuous if, and only if, it is (δ_1, δ_2)-continuous.* □

So if δ_1, δ_2 are admissible representations of topological spaces X_1, X_2, then $[\delta_1 \to \delta_2]$ is a multi-representation of the space $C(X_1, X_2)$ of all sequentially continuous partial mappings from X_1 to X_2.

The notion of admissibility and the Representation Theorem can be generalized from topological spaces to **weak limit spaces** (see [Sch02a]). In the present thesis, however, we will almost exclusively work with topological spaces that are even second-countable. If X_1 and X_2 are second-countable, then sequential continuity of f is equivalent to (topological) continuity of f (see [Wil70, Corollary 10.5.c]).

Corollary 2.7. *Let X_1, X_2 be second-countable topological spaces with admissible representations δ_1, δ_2. A partial mapping $f :\subseteq X_1 \to X_2$ is continuous if, and only if, it is (δ_1, δ_2)-continuous.* □

2.2.3 Computable T_0-spaces

The book [Wei00] introduces the notion of a *computable topological space*. The definition is modified in [GSW07], where **computable** T_0**-spaces** are defined. We agree with the authors of [GSW07] when they say: "For a foundation of Computable Topology, this new definition seems to be more useful than the former definition". A computable T_0-space is a pair[4] (X, ϑ), where X is a set and $\vartheta : \mathbb{N} \to \beta_\vartheta$ is a numbering[5] of a base $\beta_\vartheta \subseteq 2^X \setminus \{\emptyset\}$ of a T_0-topology τ_ϑ on X such that there is a computably enumerable set $B \subseteq \mathbb{N}$ with

$$(\forall\, k, \ell \in \mathbb{N})\ \Big(\vartheta(k) \cap \vartheta(\ell) = \bigcup\{\vartheta(m) \ :\ \langle k, \ell, m\rangle_{***} \in B\}\Big).$$

Let (X, ϑ) be a computable T_0-space. The **standard representation** ϑ_{std} of X is given by

$$\vartheta_{\mathrm{std}}(r) = x \quad :\Longleftrightarrow \quad [\vartheta]^{\mathrm{en}}(r) = \{U \in \beta_\vartheta \ :\ x \in U\}.$$

The standard representation is admissible. So if (Y, η) is another computable T_0-space, $[\vartheta_{\mathrm{std}} \to \eta_{\mathrm{std}}]$ is a multi-representation of the continuous partial mappings from X to Y. Three representations $\vartheta_{\mathcal{O}<}$, $\vartheta_{\mathcal{O}>}$, $\vartheta_{\mathcal{O}}$ of the topology τ_ϑ are given by

$$\begin{aligned}
\vartheta_{\mathcal{O}<}(r) = U \quad &:\Longleftrightarrow \quad U = \bigcup\{\vartheta(n) \ :\ n \in [\nu_{\mathbb{N}}]^{\mathrm{en}}(r)\},\\
\vartheta_{\mathcal{O}>}(r) = U \quad &:\Longleftrightarrow \quad [\vartheta]^{\mathrm{en}}(r) = \{V \in \beta_\vartheta \ :\ V \setminus U \neq \emptyset\},\\
\vartheta_{\mathcal{O}}\langle r, s\rangle_{\omega\omega} = U \quad &:\Longleftrightarrow \quad \vartheta_{\mathcal{O}<}(r) = \vartheta_{\mathcal{O}>}(s) = U.
\end{aligned}$$

[4] In [GSW07], an effective T_0-space is a 4-tuple: In addition to X and ϑ, the range β_ϑ of ϑ and the topology τ_ϑ generated by it are included explicitly.

[5] In [GSW07], ϑ is not required to be total; instead it is required that $\mathrm{dom}(\vartheta)$ is decidable. This, however, does not make an essential difference.

Let τ_ϑ^c be the set of closed (with respect to the topology τ_ϑ) subsets of X. Representations $\vartheta_{\mathcal{C}>}$, $\vartheta_{\mathcal{C}<}$, $\vartheta_{\mathcal{C}}$ of τ_ϑ^c are given by

$$\begin{aligned} \vartheta_{\mathcal{C}>}(r) = A \quad &:\Longleftrightarrow \quad \vartheta_{\mathcal{O}<}(r) = X \setminus A, \\ \vartheta_{\mathcal{C}<}(r) = A \quad &:\Longleftrightarrow \quad \vartheta_{\mathcal{O}>}(r) = X \setminus A, \\ \vartheta_{\mathcal{C}}(r) = A \quad &:\Longleftrightarrow \quad \vartheta_{\mathcal{O}}(r) = X \setminus A. \end{aligned}$$

Multi-representations $\vartheta_{\mathcal{K}>}$, $\vartheta_{\mathcal{K}}$ of the set τ_ϑ^κ of all compact (with respect to the topology τ_ϑ) subsets of X are given by

$$\vartheta_{\mathcal{K}>}(r) \ni K \quad :\Longleftrightarrow \quad [[\vartheta]^{<\omega}]^{\mathrm{en}}(r) = \left\{ (U_1, \ldots, U_\ell) \in \beta_\vartheta^{<\omega} \;:\; K \subseteq \bigcup_{i=1}^{\ell} U_i \right\}$$

and

$$\vartheta_{\mathcal{K}}\langle r, s\rangle_{\omega\omega} \ni K \quad :\Longleftrightarrow \quad \vartheta_{\mathcal{K}>}(r) \ni K \quad \text{and} \quad \vartheta_{\mathcal{C}<}(s) = \overline{K}.$$

These representations are single-valued[6] if τ_ϑ is T_1.

We quote a number of useful facts from [GSW07, Lemma 3.3]. We will use them without mention in the following:

Lemma 2.8. *Let (X, ϑ) be a computable T_0-space.*

(1) $\vartheta \leq \vartheta_{\mathcal{O}<}$.

(2) *$U \mapsto X \setminus U$ for open U is $(\vartheta_{\mathcal{O}<}, \vartheta_{\mathcal{C}>})$-computable. $A \mapsto X \setminus A$ for closed A is $(\vartheta_{\mathcal{C}>}, \vartheta_{\mathcal{O}<})$-computable.*

(3) *"$x \in U$" is $[\vartheta_{\mathrm{std}}, \vartheta_{\mathcal{O}<}]$-c.e.*

(4) *Union and intersection on τ_ϑ are $([\vartheta_{\mathcal{O}<}, \vartheta_{\mathcal{O}<}], \vartheta_{\mathcal{O}<})$-computable.*

(5) *Countable union on τ_ϑ is $([\vartheta_{\mathcal{O}<}]^\omega, \vartheta_{\mathcal{O}<})$-computable.*

(6) *Union and intersection on τ_ϑ^c are $([\vartheta_{\mathcal{C}>}, \vartheta_{\mathcal{C}>}], \vartheta_{\mathcal{C}>})$-computable.*

(7) *Countable intersection on τ_ϑ^c is $([\vartheta_{\mathcal{C}>}]^\omega, \vartheta_{\mathcal{C}>})$-computable.*

□

Another useful observation is the following:

[6] To see that $\vartheta_{\mathcal{K}}$ is not single-valued in general, consider the set $\{0, 1, 2\}$ with the T_0-topology $\{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}\}$. Then the sets $\{0, 2\}$ and $\{0, 1, 2\}$ are trivially compact. They also have the same closure and the same open covers.

Lemma 2.9 (Effective openness of ϑ_{std})**.** *Let* (X, ϑ) *be a computable* T_0*-space. The mapping* $w \mapsto \vartheta_{\text{std}}(w\mathbb{N}^\omega)$ *for* $w \in \mathbb{N}^{<\omega}$ *is* $([\nu_\mathbb{N}]^{<\omega}, \vartheta_{\mathcal{O}<})$*-computable.*

Proof. Expanding the definition of ϑ_{std}, we get that any $p \in \mathbb{N}^\omega$ is a valid ϑ_{std}-name of some $x \in X$ if, and only if, for every $n \in \mathbb{N}$, there is a $q \in \mathbb{N}^\omega$ such that

$$1 \in \text{range}(q) \qquad \Longleftrightarrow \qquad x \in \vartheta(n)$$

and the set

$$\{v \in \mathbb{N} \; : \; \langle n, v\rangle_{**} \in \text{range}(p)\}$$

consists of α^ω-code numbers (cf. equation (2.1)) of infinitely many basic open balls containing of q. Now suppose that an input

$$w =: (\langle r_0, s_0\rangle_{**}, \ldots, \langle r_{n-1}, s_{n-1}\rangle_{**})$$

is given. First decide whether

$$r_i = r_j \implies \alpha^\omega(s_i) \cap \alpha^\omega(s_j) \neq \emptyset.$$

If this conditions fails, w can syntactically not be prefix of a valid ϑ_{std}-name; so put out a $\vartheta_{\mathcal{O}<}$-name of $\emptyset$. Otherwise, for any $x \in X$, w can be extended to a valid ϑ_{std}-name of x if, and only if,

$$x \in \bigcap\{\vartheta(r_i) \; : \; 0 \leq i \leq n-1, \; (\forall\, q \in \alpha^\omega(s_i))(1 \in \text{range}(q))\}.$$

So this set is equal to $\vartheta_{\text{std}}(w\mathbb{N}^\omega)$. We can clearly $\vartheta_{\mathcal{O}<}$-compute and put out this set. $\square$

We will need the following fact, which we have not found in the literature:

Lemma 2.10 (Closed subsets of compact sets)**.** *Let* (X, ϑ) *be a computable* T_0*-space. Then* $(A, K) \mapsto A$ *for* $A \in \tau_\vartheta^c$ *and* $K \in \tau_\vartheta^\kappa$ *with* $A \subseteq K$ *is* $([\vartheta_\mathcal{C}, \vartheta_{\mathcal{K}>}], \vartheta_\mathcal{K})$*-computable.*

Proof. It remains to compute the $\vartheta_{\mathcal{K}>}$-information on A. It is sufficient to demonstrate how to semidecide (relative to the input information)

$$A \subseteq U_1 \cup \cdots \cup U_n \tag{2.2}$$

for $U_1, \ldots, U_n \in \beta_\vartheta$ given in ϑ-encoding. Via the $\vartheta_{C>}$-portion of the ϑ_C-name of A, we are provided with a sequence $(V_i)_i \in \beta_\vartheta^\omega$ that exhausts $X \setminus A$. As $A \subseteq K$ and K is compact, we have that (2.2) is equivalent to

$$(\exists\, \ell \in \mathbb{N})(K \subseteq U_1 \cup \cdots \cup U_n \cup V_0 \cup \cdots \cup V_\ell).$$

For each ℓ, the condition

$$K \subseteq U_1 \cup \cdots \cup U_n \cup V_0 \cup \cdots \cup V_\ell$$

can be semidecided from the given $\vartheta_{\mathcal{K}>}$-information on K. It remains to run the corresponding procedure simultaneously for all ℓ. $\square$

Lemma 2.11 (Preimages of open sets and images of compact sets)**.** *Let (X, ϑ), (Y, η) be computable T_0-spaces.*

(1) The multi-mapping

$$\mathrm{preimg} : C(X, Y) \times \tau_\eta \rightrightarrows \tau_\vartheta$$

with

$$\mathrm{preimg}(f, U) \ni V \quad :\Longleftrightarrow \quad f^{-1}(U) = V \cap \mathrm{dom}(f)$$

is $([[\vartheta_{\mathrm{std}} \to \eta_{\mathrm{std}}], \eta_{\mathcal{O}<}], \vartheta_{\mathcal{O}<})$-computable.

(2) The mapping $(f, K) \mapsto f(K)$ for compact K and total functions $f \in C(X, Y)$ is $([[\vartheta_{\mathrm{std}} \to \eta_{\mathrm{std}}], \vartheta_{\mathcal{K}}], \eta_{\mathcal{K}})$-computable.

Proof. For item (1), see [GWX08, Lemma 4.3, Theorem 4.6]. Item (2) is a generalization of [Wei03, Theorem 3.3]. We give a proof as we have not found it in the literature in this explicit form: It follows directly from [GWX08, Lemma 4.4] and the definition of $\vartheta_{\mathcal{K}}$ that $(f, K) \mapsto \overline{f(K)}$ is $([[\vartheta_{\mathrm{std}} \to \eta_{\mathrm{std}}], \vartheta_{\mathcal{K}}], \eta_{\mathcal{C}<})$-computable. So it remains to demonstrate how to semidecide (relative to the input information)

$$f(K) \subseteq U_1 \cup \cdots \cup U_n \tag{2.3}$$

for any $U_1, \ldots, U_n \in \beta_\eta$ given in η-encoding. Condition (2.3) is equivalent to

$$K \subseteq f^{-1}(U_1 \cup \cdots \cup U_n). \tag{2.4}$$

By item (1) of this lemma we can compute a sequence $(V_i)_i \in \beta_\vartheta^\omega$ with

$$f^{-1}(U_1 \cup \cdots \cup U_n) = \bigcup_{i \in \mathbb{N}} V_i.$$

By the compactness of K, condition (2.4) is equivalent to

$$(\exists\, \ell \in \mathbb{N})\ (K \subseteq V_0 \cup \cdots \cup V_\ell).$$

For each ℓ the condition

$$K \subseteq V_0 \cup \cdots \cup V_\ell$$

can be semidecided from the given $\vartheta_{\mathcal{K}}$-information on K. It remains to run the corresponding procedure simultaneously for all ℓ. □

2.2.4 Representations of $\mathbb{R}$

Let $\nu_{\mathbb{Q}}$ be some canonical total numbering of $\mathbb{Q}$.

We introduce the fundamental representation $\rho_{\mathbb{R}}$ of $\mathbb{R}$, which is admissible with respect to the standard metric topology on $\mathbb{R}$. To this end, call a sequence $(x_i)_i$ of elements of a metric space (M, d) **converging rapidly** to some $x \in M$ if

$$\lim_{i\to\infty} x_i = x \quad \text{and} \quad (\forall\, i, j \in \mathbb{N})\ (i \leq j \implies d(x_i, x_j) \leq 2^{-i}).$$

We define

$$\rho_{\mathbb{R}}(r) = x \quad :\Longleftrightarrow \quad [\nu_{\mathbb{Q}}]^{\omega}(r) \text{ converges rapidly to } x.$$

A real number x is called **computable** if it is $\rho_{\mathbb{R}}$-computable.

Put $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, $\underline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$. A representation $\overline{\rho}_{\mathbb{R}<}$ of $\overline{\mathbb{R}}$ is given by

$$\overline{\rho}_{\mathbb{R}<}(r) = x \quad :\Longleftrightarrow \quad x = \sup(\text{range}([\nu_{\mathbb{Q}}]^{\omega}(r))).$$

A representation $\underline{\rho}_{\mathbb{R}>}$ of $\underline{\mathbb{R}}$ is given by

$$\underline{\rho}_{\mathbb{R}>}(r) = x \quad :\Longleftrightarrow \quad x = \inf(\text{range}([\nu_{\mathbb{Q}}]^{\omega}(r))).$$

Put $\rho_{\mathbb{R}<} := \overline{\rho}_{\mathbb{R}<}|^{\mathbb{R}}$, and $\rho_{\mathbb{R}>} := \underline{\rho}_{\mathbb{R}>}|^{\mathbb{R}}$. A real number is called **lower semicomputable** or **upper semicomputable** if it is $\rho_{\mathbb{R}<}$-computable or $\rho_{\mathbb{R}>}$-computable, respectively.

The representation $\rho_{\mathbb{R}}$ is stronger than both $\rho_{\mathbb{R}<}$ and $\rho_{\mathbb{R}>}$. On the other hand, a $\rho_{\mathbb{R}}$-name of a real number can be computed if both a $\rho_{\mathbb{R}<}$- and $\rho_{\mathbb{R}>}$-name of that number are available. The set

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} \ :\ x < y\}$$

is $[\rho_{\mathbb{R}>}, \rho_{\mathbb{R}<}]$-c.e. The set

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} \ :\ x > y\}$$

is $[\rho_{\mathbb{R}<}, \rho_{\mathbb{R}>}]$-c.e. So both sets are $[\rho_{\mathbb{R}}, \rho_{\mathbb{R}}]$-c.e. These facts will be used implicitly in many proofs.

2.3 Computing on metric and regular spaces

2.3.1 Computable metric spaces

A **computable metric space** is a triple (X, d, α) such that (X, d) is a metric space, $\alpha : \mathbb{N} \to \mathcal{R}_\alpha$ is a numbering of a dense subset $\mathcal{R}_\alpha$ of X, and $d|_{\mathcal{R}_\alpha \times \mathcal{R}_\alpha}$ is $([\alpha, \alpha], \rho_{\mathbb{R}})$-computable. The **Cauchy representation** α_{Cauchy} of X is then given by

$$\alpha_{\text{Cauchy}}(r) = x \quad :\Longleftrightarrow \quad [\alpha]^{\omega}(r) \text{ converges rapidly to } x.$$

It is easy to see that d is $([\alpha_{\text{Cauchy}}, \alpha_{\text{Cauchy}}], \rho_{\mathbb{R}})$-computable.

If (X, d) is a metric space, put

$$B(x,s) := \{y \in X \ : \ d(x,y) < s\}, \qquad x \in X, \ s \in (0;\infty),$$

and

$$\overline{B}(x,s) := \{y \in X \ : \ d(x,y) \leq s\}, \qquad x \in X, \ s \in [0;\infty).$$

Let (X, d, α) be a computable metric space. We may define (in a canonical way) a total numbering ϑ^α of the set

$$\{B(x,s) \ : \ x \in \mathcal{R}_\alpha, \ s \in \mathbb{Q} \cap (0;\infty)\}.$$

The only things we need to know on ϑ^α are:

(1) There is an $[\alpha, \nu_{\mathbb{Q}}]^\omega$-computable sequence $(x_n, r_n)_n$ such that $\vartheta^\alpha(n) = B(x_n, r_n)$ for every $n \in \mathbb{N}$.

(2) The mapping $(a, s) \mapsto B(a, s)$ for $a \in \mathcal{R}_\alpha$ and $q \in \mathbb{Q}$, $q > 0$, is $([\alpha, \nu_{\mathbb{Q}}], \vartheta^\alpha)$-computable.

It is easy to prove the following:

Lemma 2.12 (Metric spaces as topological spaces)**.** *If (X, d, α) is a computable metric space, then (X, ϑ^α) is a computable T_0-space and $\alpha_{\text{Cauchy}} \equiv \vartheta^\alpha_{\text{std}}$.* □

Note that $(\mathbb{R}, d_{|\cdot|}, \nu_{\mathbb{Q}})$ is a computable metric space, and that

$$\vartheta^{\nu_{\mathbb{Q}}}_{\text{std}} \equiv (\nu_{\mathbb{Q}})_{\text{Cauchy}} = \rho_{\mathbb{R}}.$$

In the following, we will use this fact without further mention.

We make two observations on computable topology on metric spaces:

Lemma 2.13. *Let (X, d, α) be a computable metric space.*

(1) The map $(x, s) \mapsto \overline{B}(x, s)$ for $x \in X$ and $s \in [0;\infty)$ is $([\alpha_{\text{Cauchy}}, \rho_{\mathbb{R}}], \vartheta^\alpha_{\mathcal{C}>})$-computable.

(2) $\vartheta^\alpha_{\mathcal{K}} \leq \vartheta^\alpha_{\mathcal{C}}$.

Proof. For item (1), we need to demonstrate how to $\vartheta_{\mathcal{O}<}$-compute the set $X \setminus \overline{B}(x, s)$ from (names of) x and s. By type conversion, we can $[\vartheta_{\text{std}} \to \rho_{\mathbb{R}}]$-compute the function $d(x, \cdot)$. We can also $\vartheta^{\nu_{\mathbb{Q}}}_{\mathcal{O}<}$-compute the set $(s;\infty)$. Now note that

$$X \setminus \overline{B}(x,s) = d(x,\cdot)^{-1}((s;\infty))$$

and apply Lemma 2.11.1.

Item (2) follows from results in [BP03] (Proposition 4.2.2, Theorem 4.8.2, and Corollary 3.12.1) if one notes that their representations $\delta^{<}|^{\mathcal{K}} \sqcap \delta_{\text{cover}}$ and $\delta^{<} \sqcap \delta_{\text{union}}$ correspond to our representations $\vartheta^\alpha_{\mathcal{K}}$ and $\vartheta^\alpha_{\mathcal{C}}$, respectively. □

Lemma 2.14 (Maximization and minimization). *(1) Both the maps $K \mapsto \max K$ and $K \mapsto \min K$ for compact subsets K of $\mathbb{R}$ are $(\vartheta_{\mathcal{K}}^{\nu_{\mathbb{Q}}}, \rho_{\mathbb{R}})$-computable.*

(2) Let (X, ϑ) be a computable T_0-space. The maps $(f, K) \mapsto \max f(K)$ and $(f, K) \mapsto \min f(K)$ for compact K and continuous $f : X \to \mathbb{R}$ are $([[\vartheta_{\mathrm{std}} \to \rho_{\mathbb{R}}], \vartheta_{\mathcal{K}}], \rho_{\mathbb{R}})$-computable.

Proof. Our representations $\vartheta_{\mathcal{K}}^{\nu_{\mathbb{Q}}}$ and $\rho_{\mathbb{R}}$ are equivalent to the representations κ and ρ, respectively, used in [Wei00] (cf. [BP03, Theorem 4.8.2, Proposition 4.2] and [Wei00, Lemma 4.1.6], respectively). Item (1) is hence equivalent to [Wei00, Lemma 5.2.6.3]. Item (2) now follows by combining item (1) with Lemma 2.11.2 □

Let (X, d, α), (Y, d', α') be computable metric spaces. A mapping $f : X \to Y$ is uniformly continuous if, and only if, there exists a mapping $m : \mathbb{N} \to \mathbb{N}$ such that

$$(\forall\, n \in \mathbb{N})(\forall\, x, y \in X)\ (d(x, y) < 2^{-m(n)} \implies d'(f(x), f(y)) < 2^{-n}).$$

Such an m is called a **modulus of uniform continuity** of f. A representation $[\alpha_{\text{Cauchy}} \to \alpha'_{\text{Cauchy}}]_{\text{uni}}$ of the space $C(X, Y)_{\text{uni}}$ of all uniformly continuous partial mappings from X to Y is given by

$$[\alpha_{\text{Cauchy}} \to \alpha'_{\text{Cauchy}}]_{\text{uni}} \langle p, q \rangle_{\omega\omega} \ni f \quad :\Longleftrightarrow \quad \begin{bmatrix} [\alpha_{\text{Cauchy}} \to \alpha'_{\text{Cauchy}}](p) \ni f, \\ [\nu_{\mathbb{N}} \to \nu_{\mathbb{N}}]_{\mathrm{TOT}}(q) \text{ is a modulus of} \\ \text{uniform continuity of } f \end{bmatrix}$$

(cf. [Wei00, Definition 6.2.6.2]).

2.3.2 Computably regular spaces and metrization

Let (X, ϑ) be a computable T_0-space. Define

$$\mathrm{reg} :\subseteq \beta_\vartheta \rightrightarrows (\beta_\vartheta \times \tau_\vartheta^c)^\omega$$

by

$$\mathrm{reg}(U) \ni (V_n, A_n)_n \quad :\Longleftrightarrow \quad \bigcup_{n \in \mathbb{N}} V_n = U \quad \text{and} \quad (\forall\, n \in \mathbb{N})\ (V_n \subseteq A_n \subseteq U).$$

The topology τ_ϑ is regular in the classical sense[7] if, and only if, reg is total. If reg is in addition $(\vartheta, [\vartheta, \vartheta_{\mathcal{C}>}]^\omega)$-computable, then (X, ϑ) is called **computably regular**. This definition is due to [Sch98, GSW07].

Computable metric spaces are computably regular in the following sense:

[7]For every $x \in X$ and every $A \in \tau_\vartheta^c$ with $x \notin A$ there are $U, V \in \tau_\vartheta$ with $x \in U$, $A \subseteq V$, and $U \cap V = \emptyset$.

Lemma 2.15. *(Regularity of metric spaces) Let (X, d, α) be a computable metric space. Then the computable T_0-space (X, ϑ^α) is computably regular.*

Proof. Given an ϑ^α-name of some $U \in \beta_{\vartheta^\alpha}$, we need to compute a $[\vartheta^\alpha, \vartheta^\alpha_{\mathcal{C}>}]^\omega$-name of an element $(V_n, A_n)_n$ of $\operatorname{reg}(U)$. First, $[\alpha, \nu_{\mathbb{Q}}]$-compute (a, s) such that $U = B(a, s)$. Now choose

$$V_n := B(a, (1 - 2^{-(n+1)})s), \ A_n := \overline{B}(a, (1 - 2^{-(n+1)})s), \qquad n \in \mathbb{N},$$

and note that $(V_n, A_n)_n \in \operatorname{reg}(U)$. We can $[\vartheta^\alpha, \vartheta^\alpha_{\mathcal{C}>}]^\omega$-compute $(V_n, A_n)_n$ in view of Lemma 2.13.1. □

The next result (see [GSW07, Theorem 6.12]) is about the converse direction:

Theorem 2.16 (Metrization). *Let (X, ϑ) be a computably regular computable T_0-space. There exists a computable metric space $(\widetilde{X}, d, \alpha)$ and an injective mapping $\operatorname{emb} : X \to \widetilde{X}$ which is $(\vartheta_{\mathrm{std}}, \alpha_{\mathrm{Cauchy}})$-computable and the partial inverse $\operatorname{emb}^{-1} :\subseteq \widetilde{X} \to X$ of which is $(\alpha_{\mathrm{Cauchy}}, \vartheta_{\mathrm{std}})$-computable.* □

2.4 Computing on Banach spaces

In the following, let $\mathbb{F}$ be an element of $\{\mathbb{R}, \mathbb{C}\}$. Put

$$Q := \begin{cases} \mathbb{Q} & \text{if } \mathbb{F} = \mathbb{R}, \\ \mathbb{Q}[\mathrm{i}] & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

Here, $\mathbb{Q}[\mathrm{i}]$ is the field of complex rationals. The total numbering $\nu_{\mathbb{Q}[\mathrm{i}]}$ of $\mathbb{Q}[\mathrm{i}]$ is given by

$$\nu_{\mathbb{Q}[\mathrm{i}]}\langle r, s\rangle_{**} := \nu_{\mathbb{Q}}(r) + \mathrm{i} \cdot \nu_{\mathbb{Q}}(s).$$

If $(X, \|\cdot\|)$ is a normed space over $\mathbb{F}$, F is a subfield of $\mathbb{F}$, and V is a subset of X, denote by $\operatorname{span}_F(V)$ the set of all finite linear combinations of element of V with coefficients in F. For any points $x_0, x_1 \ldots \in X$, put

$$[x_1, \ldots, x_n] := \operatorname{span}_{\mathbb{F}}\{x_1, \ldots, x_n\},$$

and

$$[x_0, x_1, \ldots] := \operatorname{cls}(\operatorname{span}_{\mathbb{F}}\{x_0, x_1, \ldots\}).$$

For every normed space $(X, \|\cdot\|)$, define

$$B_X := \{x \in X \ : \ \|x\| \leq 1\}.$$

2.4.1 Computable normed and Banach spaces

Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{F}$ and let $e : \mathbb{N} \to \mathcal{R}_e$ be a numbering of a set $\mathcal{R}_e \subseteq X$ with

$$\mathrm{cls}(\mathrm{span}_{\mathbb{F}}(\mathcal{R}_e)) = X.$$

An e with this property is also called a **complete sequence** in X. A numbering α^e of $\mathrm{span}_Q(\mathcal{R}_e)$ can be defined in a canonical way; the only properties that this numbering ought to have are the following:

(1) There exists a $[[\nu_Q]^{<\omega}]^\omega$-computable sequence $(q_{n,1}, \dots, q_{n,\sigma(n)})_n$ such that

$$\alpha^e(n) = \sum_{i=1}^{\sigma(n)} q_{n,i} e(i-1)$$

for every $n \in \mathbb{N}$.

(2) The mapping

$$(q_1, \dots, q_n) \mapsto \sum_{i=1}^{n} q_i e(i-1)$$

for $(q_1, \dots, q_n) \in Q^{<\omega}$ is $([\nu_Q]^{<\omega}, \alpha^e)$-computable.

The tuple $(X, \|\cdot\|, e)$ is a **computable normed space** (over $\mathbb{F}$) if the restriction of $\|\cdot\|$ to $\mathcal{R}_e$ is $(\alpha^e, \rho_{\mathbb{R}})$-computable. If moreover $(X, \|\cdot\|)$ is complete, then $(X, \|\cdot\|, e)$ is called a **computable Banach space**.

It is easy to see the following:

Lemma 2.17 (Normed spaces as metric spaces)**.** *If $(X, \|\cdot\|, e)$ is a computable normed space, then $(X, d_{\|\cdot\|}, \alpha^e)$ is a computable metric space.* □

For abbreviation we will write $\theta^e := \vartheta^{\alpha^e}$.

Fix an $n \in \mathbb{N}$, $n \geq 1$. Let $u^{\mathbb{F},n}$ be a canonical numbering of the standard unit vectors in $\mathbb{F}^n$. It is not hard to see that $(\mathbb{F}^n, \|\cdot\|, u^{\mathbb{F},n})$ – with $\|\cdot\|$ being the sup-norm – is a computable Banach space. For abbreviation we will write

$$\rho_{\mathbb{F}^n} := \alpha_{\mathrm{Cauchy}}^{u^{\mathbb{F},n}}$$

and[8]

$$\theta^{\mathbb{F}^n} := \theta^{u^{\mathbb{F},n}}.$$

It is easy to prove the following:

[8] Formally, the representation $\rho_{\mathbb{R}}$ defined above is not the same as the representation $\rho_{\mathbb{R}^1}$ defined here. This, however, is not really an inconsistency because the two representations are equivalent.

Lemma 2.18. *Let $(X, \|\cdot\|, e)$ be a computable normed space over $\mathbb{F}$.*

(1) *$(x, y) \mapsto x + y$ for $x, y \in X$ is $([\alpha^e_{\mathrm{Cauchy}}, \alpha^e_{\mathrm{Cauchy}}], \alpha^e_{\mathrm{Cauchy}})$-computable.*

(2) *$(s, x) \mapsto s \cdot x$ for $s \in \mathbb{F}$ and $x \in X$ is $([\rho_{\mathbb{F}}, \alpha^e_{\mathrm{Cauchy}}], \alpha^e_{\mathrm{Cauchy}})$-computable.*

(3) *$x \mapsto \|x\|$ for $x \in X$ is $(\alpha^e_{\mathrm{Cauchy}}, \rho_{\mathbb{R}})$-computable.*

□

Lemma 2.13.1 can be strengthened for normed spaces:

Lemma 2.19 (Closed balls in normed spaces)**.** *Let $(X, \|\cdot\|, e)$ be a computable normed space. The map $(x, s) \mapsto \overline{B}(x, s)$ for $x \in X$ and $s \in [0; \infty)$ is $([\alpha^e_{\mathrm{Cauchy}}, \rho_{\mathbb{R}}], \theta^e_{\mathcal{C}})$-computable.*

Proof. Given (names of) x and s we can use type conversion to $[\alpha^e_{\mathrm{Cauchy}} \to \rho_{\mathbb{R}}]$-compute the distance function

$$y \mapsto \max(0, s - \|x - y\|)$$

of $\overline{B}(x, s)$. A $\theta^e_{\mathcal{C}}$-name of $\overline{B}(x, s)$ can then be computed by [BP03, Corollary 3.12.1]. □

2.4.2 Effective independence

For every linear space X over $\mathbb{F}$, define

$$\mathrm{IND}_X := \{(x_1, \ldots, x_n) \in X^{<\omega} \ : \ n \geq 1, \ x_1, \ldots, x_n \text{ linearly independent}\}.$$

If $(x_1, \ldots, x_n) \in \mathrm{IND}_X$, there is a unique vector $(f_1, \ldots, f_n)$ of **coordinate functionals**: The domain of each $f_i :\subseteq X \to \mathbb{F}$ is $[x_1, \ldots, x_n]$, and the f_i are uniquely defined by the condition

$$(\forall\, x \in [x_1, \ldots, x_n]) \ \left(x = \sum_{i=1}^{n} f_i(x) x_i \right).$$

Lemma 2.20 (Coordinates and unit balls in finite dimensional subspaces)**.** *Let $(X, \|\cdot\|, e)$ be a computable normed space over $\mathbb{F}$.*

(1) *The mapping*

$$(x_1, \ldots, x_n) \mapsto (f_1, \ldots, f_n)$$

that maps $(x_1, \ldots, x_n) \in \mathrm{IND}_X$ to the corresponding vector $(f_1, \ldots, f_n)$ of coordinate functionals is $([\alpha^e_{\mathrm{Cauchy}}]^{<\omega}, [\alpha^e_{\mathrm{Cauchy}} \to \rho_{\mathbb{F}}]^{<\omega})$-computable.

(2) *The mapping*

$$(x_1, \dots, x_n) \mapsto B_{[x_1,\dots,x_n]}$$

for $(x_1, \dots, x_n) \in \mathrm{IND}_X$ *is* $([\alpha^e_{\text{Cauchy}}]^{<\omega}, \theta^e_{\mathcal{K}})$*-computable.*[9]

Proof. Given α^e_{Cauchy}-names of linearly independent $x_1, \dots, x_n$. By type conversion, we can $[\rho_{\mathbb{F}^n} \to \alpha^e_{\text{Cauchy}}]$-compute the mappings

$$(\alpha_1, \dots, \alpha_n) \mapsto x_i + \sum_{\substack{1 \le j \le n \\ j \ne i}} \alpha_j x_j$$

for $i = 1, \dots, n$. It is furthermore easy to see that we can $\theta^{\mathbb{F}^n}_{\mathcal{K}}$-compute the set

$$\{(\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n \ : \ (\forall\, 1 \le i \le n)\ (|\alpha_i| \le 1)\}.$$

By Lemma 2.11.2, we can then $\theta^e_{\mathcal{K}}$-compute the sets

$$C_i := \Big\{x_i + \sum_{\substack{1 \le j \le n \\ j \ne i}} \alpha_j x_j \ : \ \alpha_j \in \mathbb{F},\, |\alpha_j| \le 1\Big\}, \qquad i = 1, \dots, n.$$

By Lemma 2.14.2, we can $\rho_{\mathbb{R}}$-compute the minimum value M that $\|\cdot\|$ obtains on the sets $C_1, \dots, C_n$. The linear independence of $x_1, \dots, x_n$ yields $M > 0$.

Let $x \in [x_1, \dots, x_n] \setminus \{0\}$ be arbitrary, say

$$x = \sum_{i=1}^n \alpha_i x_i.$$

Let $\ell \in \{1, \dots, n\}$ be such that

$$|\alpha_\ell| = \max_{1 \le i \le n} |\alpha_i|.$$

Then

$$\sum_{i=1}^n \frac{\alpha_i}{\alpha_\ell} x_i \in C_\ell,$$

which implies

$$M \le \left\| \sum_{i=1}^n \frac{\alpha_i}{\alpha_\ell} x_i \right\|.$$

For any $1 \le j \le n$ we have

$$\begin{aligned} |f_j(x)| &= \left| f_j \left(\sum_{i=1}^n \alpha_i x_i \right) \right| = |\alpha_j| \le |\alpha_\ell| \le |\alpha_\ell| M^{-1} \left\| \sum_{i=1}^n \frac{\alpha_i}{\alpha_\ell} x_i \right\| = M^{-1} \left\| \sum_{i=1}^n \alpha_i x_i \right\| \\ &= M^{-1} \|x\|. \end{aligned}$$

[9]Here and in the following, we consider balls of the form $B_{[x_1,\dots,x_n]}$ as subsets of X rather than of $[x_1, \dots, x_n]$.

As x was arbitrary, we have $\|f_j\| \leq 1/M$ for the operator norm $\|f_j\|$ of f_j.

It is sufficient to demonstrate how to compute f_j on any given $x \in [x_1, \ldots, x_n]$ up to precision 2^{-k} for any given $k \in \mathbb{N}$. This can be done by using exhaustive search to find $\alpha_1, \ldots, \alpha_n \in Q$ such that

$$\left\| x - \sum_{i=1}^{n} \alpha_i x_i \right\| < M2^{-k}.$$

Then

$$\|f_j(x) - \alpha_j\| = \left\| f_j \left(x - \sum_{i=1}^{n} \alpha_i x_i \right) \right\| < 2^{-k}.$$

We have proved item (1).

In order to see item (2), note that $B_{[x_1,\ldots,x_n]} = B_X \cap K$, where

$$K := \Big\{ \sum_{i=1}^{n} \alpha_i x_i \; : \; \alpha_i \in \mathbb{F}, \; |\alpha_i| \leq 1/M \Big\}.$$

By similar arguments as for the C_i above, we can $\theta^e_{\mathcal{K}}$-compute K. In view of Lemma 2.10, it is sufficient to show how to $\theta^e_{\mathcal{C}}$-compute $B_X \cap K$. B_X is $\theta^e_{\mathcal{C}}$-computable by Lemma 2.19. As we have a $\theta^e_{\mathcal{K}}$-name of K, we can also compute a $\theta^e_{\mathcal{C}}$-name of K by Lemma 2.13.2. Lemma 2.8 then yields that we can compute a $\theta^e_{\mathcal{C}>}$-name of $B_X \cap K$. It remains to compute a $\theta^e_{\mathcal{C}<}$-name of $B_X \cap K$. It is sufficient to demonstrate how to semidecide

$$B(a, s) \cap B_X \cap K \neq \emptyset \tag{2.5}$$

for any given $a \in \mathcal{R}_{\alpha^e}$ and $s \in \mathbb{Q} \cap (0; \infty)$. Condition (2.5) is equivalent to the existence of rationals $q_1, \ldots, q_n \in Q^n$ with

$$(\forall\, 1 \leq i \leq n)\; (|q_i| < 1/M), \qquad \left\| \sum_{i=1}^{n} q_i x_i \right\| < 1, \quad \text{and} \quad \left\| \sum_{i=1}^{n} q_i x_i - a \right\| < s.$$

Such $q_1, \ldots, q_n$ can be searched for effectively. □

Lemma 2.21 (Independence is semidecidable)**.** *Let $(X, \|\cdot\|, e)$ be a computable normed space over $\mathbb{F}$. IND_X is $[\alpha^e_{\text{Cauchy}}]^{<\omega}$-c.e.*

Proof. Given a vector $(x_1, \ldots, x_n) \in X^{<\omega}$, we need to semidecide whether the vector is in IND_X. If $n = 0$, the vector is not in IND_X. If $n = 1$, we simply semidecide $\|x_1\| > 0$. If $n > 1$ the procedure can be reduced to the procedure for $n-1$: First semidecide whether $x_1, \ldots, x_{n-1}$ are linearly independent and $\|x_n\| > 0$. In case this is detected, use Lemma 2.20.2 to $\theta^e_{\mathcal{K}}$-compute $B_{[x_1,\ldots,x_{n-1}]}$. We can now compute the distance between $x_n/\|x_n\|$ and $B_{[x_1,\ldots,x_{n-1}]}$ by means of Lemma 2.14.2 (in connection with type-conversion). It remains to semidecide whether this distance is positive. □

We prove a uniform version of the *Effective Independence Lemma* from [PER89]:

Proposition 2.22 (Effective Independence Lemma)**.** *Let* $(X, \|\cdot\|, e)$ *be a computable Banach space over* $\mathbb{F}$*. Define the partial multi-mapping* $\mathrm{EIL} :\subseteq X^\omega \rightrightarrows \mathbb{N}^\omega$ *by the condition that* $\mathrm{EIL}((x_n)_n) \ni (n_m)_m$ *if, and only if,*

$$\dim([x_0, x_1, \ldots]) = \infty \tag{2.6}$$

and $x_{n_0}, x_{n_1}, \ldots$ *are linearly independent with*

$$[x_{n_0}, x_{n_1}, \ldots] = [x_0, x_1, \ldots]. \tag{2.7}$$

EIL *is* $([\alpha^e_{\mathrm{Cauchy}}]^\omega, [\nu_\mathbb{N}]^\omega]$*-computable.*

Proof. Let an input sequence $(x_n)_n$ fulfilling (2.6) be given. By semideciding the conditions $\|x_n\| > 0$ in parallel for all n, we can compute a sequence $(\tilde{n}_m)_m$ such that

$$\{x_n \,:\, n \in \mathbb{N}\} \setminus \{0\} = \{x_{\tilde{n}_m} \,:\, m \in \mathbb{N}\}.$$

So we can assume w.l.o.g. that $x_n \neq 0$ for all n.

A sequence $(n_m)_m \in \mathrm{EIL}((x_n)_n)$ can be computed recursively: Start with $n_0 = 0$. Now suppose that $n_0, \ldots, n_k$ have already been computed; we demonstrate how to compute n_{k+1}: By Lemma 2.20.2, we can $\theta^e_\mathcal{K}$-compute $B_{[x_{n_0}, \ldots, x_{n_k}]}$, and so we can apply Lemma 2.14 to $[\rho_\mathbb{R}]^\omega$-compute the sequence $(d^{(k)}_m)_m$, where $d^{(k)}_m$ is the distance between $x_m/\|x_m\|$ and $B_{[x_{n_0}, \ldots, x_{n_k}]}$. For all $m, j \in \mathbb{N}$, we can semidecide both $d^{(k)}_m > 2^{-(k+j+1)}$ and $d^{(k)}_m < 2^{-(k+j)}$. We can hence compute a binary double sequence $(t^{(k)}_{m,j})_{m,j}$ such that

$$d^{(k)}_m \leq 2^{-(k+j+1)} \implies t^{(k)}_{m,j} = 0$$

and

$$d^{(k)}_m \geq 2^{-(k+j)} \implies t^{(k)}_{m,j} = 1$$

for all $m, j \in \mathbb{N}$. We finally search the triangular scheme

$$\begin{array}{l} t^{(k)}_{0,0}, \\ t^{(k)}_{0,1}, t^{(k)}_{1,1} \\ t^{(k)}_{0,2}, t^{(k)}_{1,2}, t^{(k)}_{2,2} \\ t^{(k)}_{0,3}, t^{(k)}_{1,3}, t^{(k)}_{2,3}, t^{(k)}_{3,3} \\ \vdots \end{array} \tag{2.8}$$

row by row and from left to right for the first occurrence of 1. There must be an occurrence of 1 in the scheme, because else one would have

$$(\forall\, m \in \mathbb{N})\ (d^{(k)}_m = 0),$$

which would imply

$$(\forall\, m \in \mathbb{N})\ (x_m \in [x_{n_0}, \ldots, x_{n_k}]),$$

in contradiction to (2.6). If the first 1 occurs at index, say, (m_0, j_0), put $n_{k+1} := m_0$.

We prove the correctness of the construction: First note that for each k, the index n_{k+1} is constructed such that the distance between $x_{n_{k+1}}/\|x_{n_{k+1}}\|$ and $B_{[x_{n_0},\ldots,x_{n_k}]}$ is positive. This ensures that the sequence $x_{n_0}, x_{n_1}, \ldots$ is linearly independent. Now assume that (2.7) fails. Then there is an $N \in \mathbb{N}$ such that

$$x_N \notin [x_{n_0}, x_{n_1}, \ldots]. \tag{2.9}$$

Let $d > 0$ be the distance between $x_N/\|x_N\|$ and $B_{[x_{n_0},x_{n_1},\ldots]}$. Let k be so large that

$$\{0, \ldots, N-1\} \cap \{n_0, n_1, \ldots\} = \{0, \ldots, N-1\} \cap \{n_0, n_1, \ldots, n_k\} \tag{2.10}$$

and $d \geq 2^{-k}$. Consider the construction of n_{k+1}. As

$$d_N^{(k)} \geq d \geq 2^{-k} \geq 2^{-(k+N)},$$

we have $t_{N,N}^{(k)} = 1$. Let (m_0, j_0) be the first index in (2.8) with $t_{m_0,j_0} = 1$. If $(m_0, j_0) \neq (N, N)$, then necessarily $n_{k+1} = m_0 \leq N - 1$, in contradiction to (2.10). So necessarily $(m_0, j_0) = (N, N)$, and hence $n_{k+1} = N$, in contradiction to (2.9). □

2.4.3 Representations of the space of bounded linear operators

If X and Y are normed spaces over $\mathbb{F}$, let $B(X,Y)$ denote the linear space of all continuous (we will also often say *bounded*) linear mappings from X to Y. $B(X,Y)$ shall be equipped with the usual operator norm:

$$\|F\| := \sup_{x \in B_X} \|F(x)\|, \qquad F \in B(X,Y).$$

$B(X,Y)$ is a Banach space if Y is a Banach space (see [Meg98, Theorem 1.4.8]). Put $B(X) := B(X,X)$ and $X^* := B(X,\mathbb{F})$. X^* is called the (topological) **dual space** of X.

Fix two computable normed spaces $(X, \|\cdot\|, e)$ and $(Y, \|\cdot\|, h)$. Of course,

$$\delta_{\text{ev}}^{e,h} := [\alpha_{\text{Cauchy}}^{e} \to \alpha_{\text{Cauchy}}^{h}]|^{B(X,Y)}$$

is a representation of $B(X,Y)$. Another representation $\delta_{\text{seq}}^{e,h}$ of $B(X,Y)$ is given by

$$\delta_{\text{seq}}^{e,h}\langle r,s\rangle_{\omega\omega} = F \quad :\Longleftrightarrow \quad [\alpha_{\text{Cauchy}}^{e}]^{\omega}(r) =: (x_i)_i,\ [\alpha_{\text{Cauchy}}^{h}]^{\omega}(s) = (F(x_i))_i,\ [x_0, x_1, \ldots] = X.$$

By adding information on the operator norm of the mapping, we obtain the representations $\delta_{\text{seq},\geq}^{e,h}$ with

$$\delta_{\text{seq},\geq}^{e,h}\langle r,s\rangle_{\omega,*} = F \quad :\Longleftrightarrow \quad \delta_{\text{seq}}^{e,h}(r) = F \quad \text{and} \quad \nu_{\mathbb{N}}(s) \geq \|F\|.$$

The following result is from [Bra03a, Theorem 4.3.2]:

Lemma 2.23. *Let $(X, \|\cdot\|, e)$ and $(Y, \|\cdot\|, h)$ be computable normed spaces. Then $\delta_{\mathrm{ev}}^{e,h} \equiv \delta_{\mathrm{seq},\geq}^{e,h}$.* □

If $(Y, \|\cdot\|, h) = (\mathbb{F}, |\cdot|, u^{\mathbb{F},1})$, we write for abbreviation $\delta_{\mathrm{dual}}^{e} := \delta_{\mathrm{ev}}^{e,h}$. Adding information on the operator norm yields the representation $\delta_{\mathrm{dual},=}^{e}$ with

$$\delta_{\mathrm{dual},=}^{e}\langle r, s\rangle_{\omega,\omega} = f \quad :\Longleftrightarrow \quad \delta_{\mathrm{dual}}^{e}(r) = f \quad \text{and} \quad \rho_{\mathbb{R}}(s) = \|f\|.$$

See [Bra03a] for a detailed comparison of various representations of the space of linear bounded operators.

2.4.4 Computable Hilbert spaces

A **computable Hilbert space** is a computable Banach space whose underlying Banach space is a Hilbert space.

The article [BY06] is a source for many results on computable Hilbert spaces. It is proved, for example, that there is, up to bi-computable isometric isomorphism, only one infinite-dimensional computable Hilbert space over $\mathbb{F}$, namely $(\ell_2, \|\cdot\|, e)$, where e is the standard basis of ℓ_2. In our further considerations, we will hence restrict ourselves to this canonical computable Hilbert space.

Recall that the **adjoint** $T^* \in B(Y, X)$ of a mapping $T \in B(X, Y)$, for X, Y being Hilbert spaces, is defined by

$$(\forall\, x \in X)(\forall\, y \in Y)\ (\langle Tx \mid y\rangle = \langle x \mid T^* y\rangle).$$

The mapping $T \mapsto T^*$ for $T \in B(\ell_2)$ is not $(\delta_{\mathrm{ev}}^{e,e}, \delta_{\mathrm{ev}}^{e,e})$-computable (see [BY06, Example 6.5]). This suggests to introduce the following representation of $B(\ell_2)$, that is stronger than $\delta_{\mathrm{ev}}^{e,e}$:

$$\Delta\langle r, s\rangle_{\omega\omega} = T \quad :\Longleftrightarrow \quad \delta_{\mathrm{ev}}^{e,e}(r) = T \quad \text{and} \quad \delta_{\mathrm{ev}}^{e,e}(s) = T^*.$$

2.5 Computing on locally finite Borel measures

2.5.1 The standard representations

We assume that the reader is familiar with the basic notions of probability and measure theory (see e.g. [VTC87, Kal02]).

If X is a topological space, denote by $\mathcal{B}(X)$ the Borel σ-algebra on X. A measure ν on $\mathcal{B}(X)$ is called **locally finite** if every $x \in X$ has an open neighbourhood U with $\nu(U) < \infty$.

Lemma 2.24. *A locally finite measure ν on the Borel sets $\mathcal{B}(X)$ of a topological Lindelöf space X is uniquely defined by its restriction to open subsets.*

Proof. Let ν_1 and ν_2 be locally finite Borel measures on X that coincide on all open subsets. By the local finiteness of ν_1, ν_2 and the Lindelöf property of X, there exists an ascending sequence $(O_n)_n$ of open sets with $\bigcup_n O_n = X$ and $\nu_1(O_n) = \nu_2(O_n) < \infty$ for every n. Fix some n for a moment. Consider the trace σ-algebra

$$O_n \cap \mathcal{B}(X) := \{O_n \cap B \ : \ B \in \mathcal{B}(X)\}.$$

$O_n \cap \mathcal{B}(X)$ is generated by

$$\mathcal{B}_n := \{U \in 2^X \ : \ U \text{ open}, \, U \subseteq O_n\}.$$

(This can be seen by noting that

$$\{B \in 2^X \ : \ O_n \cap B \in \sigma(\mathcal{B}_n)\}$$

is a σ-algebra that contains all open sets and hence contains $\mathcal{B}(X)$.) $\mathcal{B}_n$ is closed under finite intersections, and $\nu_1|_{\mathcal{B}_n} = \nu_2|_{\mathcal{B}_n}$. We hence have from [Kal02, Lemma 17.1] that

$$\nu_1|_{O_n \cap \mathcal{B}(X)} = \nu_2|_{O_n \cap \mathcal{B}(X)}.$$

Now let $B \in \mathcal{B}(X)$ be arbitrary. We have

$$\nu_1(O_n \cap B) = \nu_2(O_n \cap B)$$

for every n. Letting $n \to \infty$ and using continuity from below (see [Kal02, Lemma 1.14.1]) yields $\nu_1(B) = \nu_2(B)$. □

Let (X, ϑ) be a computable T_0-space, and denote by $\mathcal{M}(X)$ the class $\mathcal{M}(X)$ of locally finite Borel measures on X. It is easy to see that $\nu|_{\mathcal{O}(X)}$ is $(\vartheta_{\mathcal{O}<}, \overline{\rho}_{\mathbb{R}<})$-continuous for every $\nu \in \mathcal{M}(X)$. This observation and the preceding lemma justify the following definition: $\vartheta_{\mathcal{M}<}$ with

$$\vartheta_{\mathcal{M}<}(r) = \nu \quad :\Longleftrightarrow \quad [\vartheta_{\mathcal{O}<} \to \overline{\rho}_{\mathbb{R}<}]_{\mathrm{TOT}}(r) = \nu|_{\mathcal{O}(X)}$$

is a representation of $\mathcal{M}(X)$.

Schröder [Sch07] studied a canonical representation of probability measures on admissibly represented topological spaces. The special case of his representation for computable topological spaces is equivalent to the restriction of our representation to probability measures. For probability measures on the unit interval, an equivalent representation was already studied in [Wei99]. We would also like to mention the work of Gács [Gác05] who defined a representation for probability measures on separable metric spaces, which is consistent with the definition that we use (see [Sch07]).

In [Wei99], we also find a representation of finite measures on $[0; 1]$ that are not necessarily probability; that representation amends the one just described by information on the number $\nu(X)$. We proceed accordingly: Let (X, ϑ) be a computable T_0-space. Then $\vartheta_{\mathcal{M}_0<}$ with

$$\vartheta_{\mathcal{M}_0<}\langle r, s\rangle_{\omega\omega} = \nu \quad :\Longleftrightarrow \quad \vartheta_{\mathcal{M}<}(r) = \nu \quad \text{and} \quad \rho_{\mathbb{R}>}(s) = \nu(X)$$

is a representation of the class $\mathcal{M}_0(X)$ of finite Borel measures on X.

The next lemma follows directly from the definitions of the involved representations and Lemma 2.11.1:

Lemma 2.25 (Image measures). *Let (X, ϑ) and (Y, η) be computable T_0-spaces. The map $(\nu, f) \mapsto \nu \circ f^{-1}$ for $\nu \in \mathcal{M}_0(X)$ and total functions $f \in C(X, Y)$ is $([\vartheta_{\mathcal{M}_0<}, [\vartheta_{\mathrm{std}} \to \eta_{\mathrm{std}}]], \eta_{\mathcal{M}_0<})$-computable.* □

2.5.2 The strong representation

We will work with a stronger representation to prove theorems on computable integration. In order to define this representation, let us first fix a canonical numbering $\nu_{\mathcal{A}}$ of the set $\mathcal{A}$ of all finite algebraic expressions involving variable symbols $\mathtt{u}_0, \mathtt{u}_1, \ldots$, the one-ary function symbol `cmpl`, and the two-ary function symbol `union`. If $w \in \mathcal{A}$ and $(A_n)_n$ is a sequence of subsets of some set X, then let $\mathrm{eval}(w, X, (A_n)_n)$ be the subset of X that is obtained from w by assigning the value A_n to the variable $\mathtt{u}_n$ for all n and interpreting `cmpl` as set complementation in X and `union` as set union.

Now let (X, ϑ) be a computable T_0-space. Define the representation $\vartheta_{\mathcal{M}_0}$ of $\mathcal{M}_0(X)$ by

$$\vartheta_{\mathcal{M}_0}\langle r, s\rangle_{\omega\omega} = \nu \quad :\Longleftrightarrow \quad \left[\begin{array}{l} [\vartheta_{\mathcal{O}<}]^{\omega}(r) =: (U_n)_n, \\ (\forall\, n \in \mathbb{N})\ \big(\vartheta(n) = \bigcup_{i\in\mathbb{N}} U_{\langle n,i\rangle_{**}}\big), \\ (\forall\, w \in \mathcal{A})\ \big([\nu_{\mathcal{A}} \to \rho_{\mathbb{R}}](s)(w) = \nu(\mathrm{eval}(w, X, (U_n)_n))\big). \end{array}\right. \tag{2.11}$$

So a $\vartheta_{\mathcal{M}_0}$-name of a measure ν encodes an *alternative base* of τ_ϑ, information on its connection to the original base β_ϑ, and the ν-contents of algebraic expressions of the alternative base's elements.

It is easy to see that $\vartheta_{\mathcal{M}_0}$ is stronger than $\vartheta_{\mathcal{M}_0<}$:

Lemma 2.26. *Let (X, ϑ) be a computable T_0-space. Then $\vartheta_{\mathcal{M}_0} \leq \vartheta_{\mathcal{M}_0<}$.* □

We will see later that $\vartheta_{\mathcal{M}_0}$ and $\vartheta_{\mathcal{M}_0<}$ are equivalent for an important class of spaces. In general, however, $\vartheta_{\mathcal{M}_0}$ is properly stronger:

Example 2.27. Suppose $X = \{0, 1\}$ and

$$\vartheta(n) = \begin{cases} \{0\} & \text{if } n = 0, \\ \{0,1\} & \text{otherwise.} \end{cases}$$

Then (X, ϑ) is a computable T_0-space. For every $s \in [0; 1]$, let ν_s be the unique Borel probability measure on X with $\nu(\{0\}) = s$. It is easy to see that $s \mapsto \nu_s$ is $(\rho_{\mathbb{R}<}, \vartheta_{\mathcal{M}_0<})$-computable and that $\nu_s \mapsto s$ is $(\vartheta_{\mathcal{M}_0}, \rho_{\mathbb{R}})$-computable. $\rho_{\mathbb{R}<} \not\leq_t \rho_{\mathbb{R}}$ thus implies $\vartheta_{\mathcal{M}_0<} \not\leq_t \vartheta_{\mathcal{M}_0}$. □

2.5.3 Computable measures on computably regular spaces

The proof of the next proposition builds on the fact that for any finite Borel measure ν on a metric space, one can exhaust every open ball by balls whose boundaries' ν-content is zero.[10]

Proposition 2.28 (Strong computability of measures on metric spaces)**.** *Let (X, d, α) be a computable metric space. Then $\vartheta^{\alpha}_{\mathcal{M}_0<} \equiv \vartheta^{\alpha}_{\mathcal{M}_0}$.*

Proof. In view of Lemma 2.26, it remains to prove $\vartheta^{\alpha}_{\mathcal{M}_0<} \leq \vartheta^{\alpha}_{\mathcal{M}_0}$. Let a $\vartheta^{\alpha}_{\mathcal{M}_0<}$-name of a measure ν be given. Put

$$\begin{aligned} D &:= \{(a, r, s) \in \mathcal{R}_\alpha \times \mathbb{Q} \times \mathbb{Q} \;:\; 0 < r \leq s\}, \\ D' &:= \{(a, r, s) \in \mathcal{R}_\alpha \times \mathbb{Q} \times \mathbb{Q} \;:\; 0 < r < s\} \end{aligned}$$

and

$$R(a, r, s) := \overline{B}(a, s) \setminus B(a, r)$$

for every $(a, r, s) \in D$. In view of Lemma 2.13.1, we can $\vartheta^{\alpha}_{\mathcal{O}<}$-compute

$$X \setminus R(a, r, s) = (X \setminus \overline{B}(a, s)) \cup B(a, r)$$

for any given $(a, r, s) \in D$; we can then $\rho_>$-compute the value

$$\nu(R(a, r, s)) = \nu(X) - \nu(X \setminus R(a, r, s)).$$

This yields that given $(a, r, s) \in D'$ and $k \in \mathbb{N}$, we can $[\nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}]$-enumerate the set

$$\begin{aligned} S(a, r, s, k) := \{(r', s') \in \mathbb{Q} \times \mathbb{Q} \;:\; & r \leq r' < s' \leq s, \\ & s' - r' < 2^{-k}, \\ & \nu(R(a, r', s')) < 2^{-k}\}. \end{aligned}$$

[10] This fact also plays are role, for example, in the proof of the classical *Portmanteau Theorem* from the theory of weak convergence of probability measures (see [Kal02]). In computable analysis, the idea appeared in [Wei99, proof of Theorem 3.6].

Let us show that this set is nonempty: Suppose otherwise. Let $M \in \mathbb{N}$ be such that $M2^{-k} > \nu(X)$. Choose arbitrary $r_1, \dots, r_M, s_1, \dots, s_M \in \mathbb{Q}$ such that

$$r < r_1 < s_1 < \dots < r_M < s_M < s \quad \text{and} \quad (\forall\, 1 \leq i \leq M)\ (s_i - r_i < 2^{-k}).$$

$S(a, r, s, k) = \emptyset$ implies

$$(\forall\, 1 \leq i \leq M)\ (\nu(R(a, r_i, s_i)) \geq 2^{-k}).$$

As the $R(a, r_i, s_i)$ are pairwise disjoint, we have

$$\nu\big(\bigcup_{i=1}^{M} R(a, r_i, s_i)\big) = \sum_{i=1}^{M} \nu(R(a, r_i, s_i)) \geq M2^{-k} > \nu(X).$$

Contradiction!

Given $(a, r, s) \in D'$ and repeatedly using exhaustive search, we are able to compute two sequences $(r_i)_i, (s_i)_i \in \mathbb{Q}^\omega$ such that

$$(r_0, s_0) \in S(a, r, s, 0) \quad \text{and} \quad (\forall\, i \in \mathbb{N})\ ((r_{i+1}, s_{i+1}) \in S(a, r_i, s_i, i + 1)).$$

$(r_i)_i$ and $(s_i)_i$ converge rapidly to the same limit, say, t. One has

$$\nu(R(a, t, t)) = \nu\big(\bigcap_{i \in \mathbb{N}} R(a, r_i, s_i)\big) = \lim_{i \to \infty} \nu(R(a, r_i, s_i)) \leq \lim_{i \to \infty} 2^{-i} = 0.$$

So far we have shown: The total multifunction $\phi : D' \rightrightarrows \mathbb{R}$ with

$$\phi(a, r, s) \ni t \qquad \Longleftrightarrow \qquad r \leq t \leq s \quad \text{and} \quad \nu(R(a, t, t)) = 0$$

is well-defined and $([\alpha, \nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}]|^{D'}, \rho_{\mathbb{R}})$-computable.

Let $(a_n, s_n)_n \in \mathcal{R}_\alpha \times \mathbb{Q}$ be an $[\alpha, \nu_{\mathbb{Q}}]^\omega$-computable sequence with $\vartheta^\alpha(n) = B(a_n, s_n)$ for every $n \in \mathbb{N}$. We can compute a function $t : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ such that

$$(\forall\, n, i \in \mathbb{N})\ (t(n, i) \in \phi(a_n, s_n(1 - 2^{-(i+1)}), s_n(1 - 2^{-(i+2)}))).$$

We hence have

$$(\forall\, n, i \in \mathbb{N})\ (0 \leq t(n, i) \leq s_n) \quad \text{and} \quad (\forall\, n \in \mathbb{N})\ \big(\lim_{i \to \infty} t(n, i) = s_n\big) \tag{2.12}$$

and

$$(\forall\, n, i \in \mathbb{N})\ (\nu(R(a_n, t(n, i), t(n, i))) = 0). \tag{2.13}$$

We $[\vartheta^\alpha_{\mathcal{O}<}]^\omega$-compute the sequences $(U_n)_n$ and $(\widetilde{U}_n)_n$ with

$$U_{\langle n,i \rangle_{**}} := B(a_n, t(n, i)) \quad \text{and} \quad \widetilde{U}_{\langle n,i \rangle_{**}} := X \setminus \overline{B}(a_n, t(n, i))$$

for all $n, i \in \mathbb{N}$. Recall that our task is to compute a $\vartheta^\alpha_{\mathcal{M}_0}$-name of ν and that the first component of such a name is an $[\vartheta^\alpha_{\mathcal{O}<}]^\omega$-name of an alternative base of the topology of X. We take $(U_n)_n$ to be this base. In fact, it then follows from (2.12) that

$$(\forall\, n \in \mathbb{N})\ \big(\vartheta^\alpha(n) = \bigcup_{i \in \mathbb{N}} U_{\langle n,i\rangle_{**}}\big),$$

so the first requirement on $(U_n)_n$ from (2.11) is fulfilled.

It remains to demonstrate how to compute the second component of the $\vartheta^\alpha_{\mathcal{M}_0}$-name of ν. To this end, it is sufficient (by type conversion) to demonstrate how to $\rho_{\mathbb{R}}$-compute the ν-contents of algebraic expressions of elements of $(U_n)_n$. Put

$$E := \{(A, V, \widetilde{V}) \in \mathcal{B}(X) \times \tau_{\vartheta^\alpha} \times \tau_{\vartheta^\alpha}\ :\ V \subseteq A,\ \widetilde{V} \subseteq X \setminus A,\ \nu(V) + \nu(\widetilde{V}) = \nu(X)\}.$$

If $(A, V, \widetilde{V}) \in E$, we can $\rho_{\mathbb{R}}$-compute $\nu(A)$ from $\vartheta^\alpha_{\mathcal{O}<}$-names of V and $\widetilde{V}$, because $\nu(A) = \nu(V)$ (which yields the $\rho_{\mathbb{R}<}$-information) and $\nu(A) = \nu(X) - \nu(\widetilde{V})$ (which yields the $\rho_{\mathbb{R}>}$-information). It is hence sufficient to demonstrate how to $\vartheta^\alpha_{\mathcal{O}<}$-compute sets $V_w, \widetilde{V}_w$ with

$$(\mathrm{eval}(w, X, (U_n)_n), V_w, \widetilde{V}_w) \in E$$

for given $w \in \mathcal{A}$. This can be done recursively: If $w = \mathtt{u}_k$ for some $k \in \mathbb{N}$, equation (2.13) yields

$$(\mathrm{eval}(w, X, (U_n)_n), U_k, \widetilde{U}_k) \in E.$$

If w has the form $\mathtt{cmpl}(w')$, first compute V, $\widetilde{V}$ with

$$(\mathrm{eval}(w', X, (U_n)_n), V, \widetilde{V}) \in E;$$

then

$$(\mathrm{eval}(w, X, (U_n)_n), \widetilde{V}, V) \in E.$$

If w has the form $\mathtt{union}(w', w'')$, first compute V', $\widetilde{V}'$, V'', $\widetilde{V}''$ with

$$(\mathrm{eval}(w', X, (U_n)_n), V', \widetilde{V}') \in E \quad \text{and} \quad (\mathrm{eval}(w'', X, (U_n)_n), V'', \widetilde{V}'') \in E;$$

then

$$(\mathrm{eval}(w, X, (U_n)_n), V' \cup V'', \widetilde{V}' \cap \widetilde{V}'') \in E.$$

□

The previous result can be generalized:

Theorem 2.29 (Strong computability of measures on regular spaces)**.** *Let (X, ϑ) be a computably regular computable T_0-space. Then $\vartheta_{\mathcal{M}_0<} \equiv \vartheta_{\mathcal{M}_0}$.*

Proof. In view of Lemma 2.26, it remains to prove $\vartheta_{\mathcal{M}_0<} \leq \vartheta_{\mathcal{M}_0}$. Let $(\widetilde{X}, d, \alpha)$ and emb be as in Theorem 2.16. Given a $\vartheta_{\mathcal{M}_0<}$-name of a measure ν, we can $\vartheta^{\alpha}_{\mathcal{M}_0<}$-compute $\nu \circ \mathrm{emb}^{-1}$ by Lemma 2.25. By Proposition 2.28, we can then even $\vartheta^{\alpha}_{\mathcal{M}_0}$-compute $\nu \circ \mathrm{emb}^{-1}$. By the computability of emb^{-1} and Lemma 2.11.1, we can $[\vartheta^{\alpha}_{\mathcal{O}<}]^{\omega}$-compute a sequence $(V_n)_n$ of open subsets of $\widetilde{X}$ such that

$$(\forall\, n \in \mathbb{N})(\mathrm{emb}(\vartheta(n)) = V_n \cap \mathrm{range}(\mathrm{emb})).$$

From the $\vartheta^{\alpha}_{\mathcal{M}_0}$-information on $\nu \circ \mathrm{emb}^{-1}$, we can $[\vartheta^{\alpha}_{\mathcal{O}<}]^{\omega}$-compute a sequence $(\widetilde{U}_n)_n$ such that

$$(\forall\, n \in \mathbb{N})\ \big(V_n = \bigcup_{i \in \mathbb{N}} \widetilde{U}_{\langle n,i\rangle_{**}}\big),$$

and such that we can compute the $(\nu \circ \mathrm{emb}^{-1})$-contents of algebraic expressions of the $\widetilde{U}_n$. By the computability of emb and Lemma 2.11.1, we can $[\vartheta_{\mathcal{O}<}]^{\omega}$-compute the sequence $(U_n)_n$ with

$$(\forall\, n \in \mathbb{N})\ (U_n = \mathrm{emb}^{-1}(\widetilde{U}_n)).$$

Then

$$(\forall\, n \in \mathbb{N})\ \big(\vartheta(n) = \bigcup_{i \in \mathbb{N}} U_{\langle n,i\rangle_{**}}\big).$$

We take (a name of) $(U_n)_n$ to be the first component of our output $\vartheta_{\mathcal{M}_0}$-name of ν. It remains to demonstrate how to compute the ν-contents of algebraic expressions of elements of $(U_n)_n$. It is easy to prove by induction on the structure of $w \in \mathcal{A}$ that

$$(\forall\, w \in \mathcal{A})\ (\mathrm{eval}(w, X, (U_n)_n) = \mathrm{emb}^{-1}(\mathrm{eval}(w, \widetilde{X}, (\widetilde{U}_n)_n))).$$

This yields

$$(\forall\, w \in \mathcal{A})\ (\nu(\mathrm{eval}(w, X, (U_n)_n)) = (\nu \circ \mathrm{emb}^{-1})(\mathrm{eval}(w, \widetilde{X}, (\widetilde{U}_n)_n))).$$

Finally, note that we can evaluate the expression on the right-hand side for any given w using the $\vartheta^{\alpha}_{\mathcal{M}_0}$-information on $\nu \circ \mathrm{emb}^{-1}$. $\square$

2.6 Computable reducibility of functions

Let M_1, M_2, M_3, M_4 be sets with representations $\delta_1, \delta_2, \delta_3, \delta_4$, and let $f :\subseteq M_1 \to M_2$ and $g :\subseteq M_3 \to M_4$ be partial mappings. One says that $(f; \delta_1, \delta_2)$ is **computably reducible** to $(g; \delta_3, \delta_4)$ and writes $(f; \delta_1, \delta_2) \leq_c (g; \delta_3, \delta_4)$ if there is a (δ_1, δ_3)-computable mapping $\mathrm{PRE} :\subseteq M_1 \to M_3$ and a $([\delta_1, \delta_4], \delta_2)$-computable mapping $\mathrm{POST} :\subseteq M_1 \times M_4 \to M_2$ such that $\mathrm{POST}(x, g(\mathrm{PRE}(x)))$ is well-defined and equal to $f(x)$ for every $x \in \mathrm{dom}(f)$.

Whenever we prove computable reducibility, we have to describe the mappings PRE and POST as above. We will informally speak of **preprocessing** and **postprocessing**, respectively.

We write $(f; \delta_1, \delta_2) \cong_c (g; \delta_3, \delta_4)$ if both $(f; \delta_1, \delta_2) \leq_c (g; \delta_3, \delta_4)$ and $(g; \delta_3, \delta_4) \leq_c (f; \delta_1, \delta_2)$. This computable reducibility notion was considered in [Bra99, Bra05, Ghe06, Bra08a]; an analogue notion of *continuous reducibility* was studied first in [Wei92].

One can easily prove the transitivity of $\leq_c$.

Let $C_1, C_2 : \mathbb{N}^\omega \to \mathbb{N}^\omega$ be defined by

$$C_1(p)(n) := \begin{cases} 0 & \text{if } (\exists\, m \in \mathbb{N})\ (p\langle n, m\rangle_{**} \neq 0), \\ 1 & \text{otherwise,} \end{cases}$$

and

$$C_2(p)(n) := \begin{cases} 0 & \text{if } (\exists\, m \in \mathbb{N})(\forall\, k \in \mathbb{N})\ (p\langle n, m, k\rangle_{***} \neq 0), \\ 1 & \text{otherwise.} \end{cases}$$

Following this pattern, one can define functions $C_3, C_4, \ldots$ (see [Bra05]), but we will only need C_1 and C_2.

Let $f :\subseteq M_1 \to M_2$ be a partial mapping of sets M_1, M_2 with representations δ_1, δ_2. For $k \in \{2, 3\}$, we call $(f; \delta_1, \delta_2)$

- Σ^0_k**-computable** if $(f; \delta_1, \delta_2) \leq_c (C_{k-1}, \mathrm{id}_{\mathbb{N}^\omega}, \mathrm{id}_{\mathbb{N}^\omega})$,
- Σ^0_k**-hard** if $(C_{k-1}, \mathrm{id}_{\mathbb{N}^\omega}, \mathrm{id}_{\mathbb{N}^\omega}) \leq_c (f; \delta_1, \delta_2)$, and
- Σ^0_k**-complete** if $(C_{k-1}, \mathrm{id}_{\mathbb{N}^\omega}, \mathrm{id}_{\mathbb{N}^\omega}) \cong_c (f; \delta_1, \delta_2)$.

Σ^0_2-hard mappings necessarily map some computable points to uncomputable points (see [Bra99, Bra05]):

Lemma 2.30 (Lack of computable invariance)**.** *Let M_1, M_2 be sets with representations δ_1, δ_2. Let $f :\subseteq M_1 \to M_2$ be a partial mapping. If $(f; \delta_1, \delta_2)$ is Σ^0_2-hard, then there exists a δ_1-computable $x \in \mathrm{dom}(f)$ such that $f(x)$ is not δ_2-computable.* □

We present the *First Main Theorem* from [PER89] in the form given in [Bra99, Theorem 4.3]. Recall that a linear mapping of topological vector spaces is called **closed** if it has a closed graph.

Theorem 2.31 (First Main Theorem)**.** *Let $(X, \|\cdot\|, e)$ and $(Y, \|\cdot\|, g)$ be computable Banach spaces, $(y_n)_n \in Y^\omega$ an $[\alpha^g_{\mathrm{Cauchy}}]^\omega$-computable sequence with $[y_0, y_1, \ldots] = Y$, and $F :\subseteq Y \to X$ a linear unbounded closed mapping such that $(Fy_n)_n$ is $[\alpha^e_{\mathrm{Cauchy}}]^\omega$-computable. Then $(F; \alpha^g_{\mathrm{Cauchy}}, \alpha^e_{\mathrm{Cauchy}})$ is Σ^0_2-hard.* □

Another useful result is the following, which is a Corollary to [Bra05, Proposition 9.1]:

Proposition 2.32 (Σ^0_2-completeness of the limit operation)**.** *Let $(X, \|\cdot\|, e)$ be a computable Banach space. Consider the map $\mathrm{LIM} :\subseteq X^\omega \to X$ with*

$$\mathrm{LIM}((x_n)_n) = x \quad :\Longleftrightarrow \quad (x_n)_n \text{ converges to } x.$$

$(\mathrm{LIM}; [\alpha^e_{\mathrm{Cauchy}}]^\omega, \alpha^e_{\mathrm{Cauchy}})$ is Σ^0_2-complete. □

Chapter 3

On the Effective Existence of Schauder Bases in Banach Spaces

3.1 Motivation

3.1.1 Schauder bases

Let X be an infinite-dimensional Banach space over $\mathbb{F}$. A sequence $(x_i)_i \in X^\omega$ is called a **Schauder basis** of X if for every $x \in X$ there exists exactly one sequence $(\alpha_i)_i \in \mathbb{F}^\omega$ such that

$$\lim_{n\to\infty} \left\| x - \sum_{i=0}^{n-1} \alpha_i x_i \right\| = 0.$$

A sequence $(x_i)_i \in X^\omega$ is called **basic** if it is a Schauder basis of the subspace $[x_0, x_1, \ldots]$.

Let X be a finite-dimensional Banach space over $\mathbb{F}$. A finite sequence $(x_1, \ldots, x_n) \in X^{<\omega}$ is a **Schauder basis** of X if for every $x \in X$ there exists exactly one tuple $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^{<\omega}$ such that

$$x = \sum_{i=1}^{n} \alpha_i x_i.$$

In this case, the notion of a Schauder basis coincides with the usual (Hamel) notion of a vector space basis. $(x_1, \ldots, x_n)$ is called **basic** if it is a Schauder basis of $[x_1, \ldots, x_n]$, i.e. if $x_1, \ldots, x_n$ are linearly independent.

We will from now on use the term *basis* synonymously with *Schauder basis*.

The following result is due to Banach (see [Meg98, Corollary 4.1.25]):

Proposition 3.1 (Characterization of basic sequences)**.** *Let X be an infinite-dimensional Banach space over $\mathbb{F}$. A sequence $(x_i)_i \in X^\omega$ is basic if, and only if,*

(1) no x_i is equal to zero, and

(2) there exists a constant $M \in \mathbb{R}$ such that

$$\|\sum_{i=0}^{m} \alpha_i x_i\| \leq M \|\sum_{i=0}^{n} \alpha_i x_i\| \tag{3.1}$$

for all $m, n \in \mathbb{N}$, $m \leq n$, and $\alpha_0, \ldots, \alpha_n \in \mathbb{F}$.

□

If $(x_i)_i \in X^\omega$ is basic, then the **basis constant** $\mathrm{bc}((x_i)_i)$ of $(x_i)_i$ is defined as the minimum M such that (3.1) holds for all $m, n \in \mathbb{N}$, $m \leq n$, and all $\alpha_0, \ldots, \alpha_n \in \mathbb{F}$. If $(x_1, \ldots, x_n) \in X^{<\omega}$ is basic, then the **basis constant** $\mathrm{bc}((x_1, \ldots, x_n))$ of $(x_1, \ldots, x_n)$ is defined as the minimum M such that

$$\|\sum_{i=1}^{m} \alpha_i x_i\| \leq M \|\sum_{i=1}^{n} \alpha_i x_i\|$$

holds for all $m \in \mathbb{N}$, $m \leq n$, and all $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$. If X is a Banach space with a basis, then the **basis constant** $\mathrm{bc}(X)$ of X is defined as the infimum over the basis constants of all bases of X.

It is obvious that any basis constant must be at least 1. A basis $(x_i)_i$ of some infinite-dimensional Banach space X is called **monotone** if $\mathrm{bc}((x_i)_i) = 1$. $(x_i)_i$ is called **shrinking** if

$$(\forall f \in X^*) \left(\lim_{n \to \infty} \sup\{|f(x)| \ : \ x \in B_{[x_n, x_{n+1}, \ldots]}\} = 0 \right).$$

Schauder bases are an important subject in the geometry of Banach spaces. See e.g. [Meg98, Sin70, Sin81] for examples of Schauder bases as well as background information and applications.

3.1.2 Schauder bases and an effective theory of compact operators

Recall that a linear mapping $F : X \to Y$ of Banach spaces X, Y is **compact** if $\overline{F(B_X)}$ is a compact subset of Y. The compact mappings from X to Y form a closed subspace $K(X, Y)$ of $B(X, Y)$ (see [Meg98, Corollary 3.4.9]).

Before we collect a number of classical facts on compact operators, let us recall a number of definitions: The **adjoint** $F^* \in B(Y^*, X^*)$ of $F \in B(X, Y)$ is given by the formula

$$F^*(g)(x) := g(F(x)), \qquad g \in Y^*,\, x \in X.$$

The **rank** of a linear mapping is the dimension of its range. Denote by $\mathrm{FR}(X, Y)$ the linear subspace of $B(X, Y)$ that consists of all bounded linear mappings of finite rank. A Banach space X has (Grothendieck's) **approximation property** (AP) if for every compact subset K of X and every $\varepsilon > 0$, there is an $L_{K,\varepsilon} \in \mathrm{FR}(X, X)$ such that

$$\sup_{x \in K} \|x - L_{K,\varepsilon}(x)\| \leq \varepsilon.$$

For the following see [Meg98, Theorem 4.1.33]:

Proposition 3.2. *Let X be a Banach space with a Schauder basis. Then X has the approximation property.* □

The following facts can all be found in [Meg98, Section 3.4]:

Proposition 3.3 (Properties of compact mappings)**.** *Let X, Y, Z be Banach spaces over $\mathbb{F}$.*

(1) Suppose $F \in B(Y, Z)$, $G \in B(X, Y)$. Then $FG \in K(X, Z)$ if F or G is compact.

(2) (Theorem of Schauder): Suppose $F \in B(X, Y)$. Then $F \in K(X, Y)$ if, and only if, $F^* \in K(Y^*, X^*)$.

(3) $\mathrm{FR}(X, Y) \subseteq K(X, Y)$.

(4) (Grothendieck): $\mathrm{FR}(X, Y)$ is dense in $K(X, Y)$ if, and only if, Y has the approximation property.

□

Brattka and Dillhage [BD07] sought (among others) for effective versions of the items of Proposition 3.3. They introduced a representation of the space of compact mappings between computable Banach spaces $(X, \|\cdot\|, e)$ and $(Y, \|\cdot\|, h)$, which combines an $[\alpha^e_{\mathrm{Cauchy}} \to \alpha^h_{\mathrm{Cauchy}}]$-name of a mapping $F \in K(X, Y)$ with a $\theta^h_{\mathcal{K}}$-name of $\overline{F(B_X)}$. Using this representation, they were able to prove that natural effective formulations of the items of Proposition 3.3 hold true. However, they worked with the additional assumption that the computable Banach spaces under consideration possess computable Schauder bases. Sometimes, it was additionally assumed that these bases are monotone and/or *computably shrinking*. Brattka and Dillhage call a basis $(x_i)_i$ of a computable Banach space $(X, \|\cdot\|, e)$ **computably shrinking** if $m : X^* \rightrightarrows \mathbb{N}^\omega$ with

$$m(f) \ni (k_n)_n \quad :\Longleftrightarrow \quad (\forall\, n \in \mathbb{N})\left(\sup\{|f(x)|\ :\ x \in B_{[x_{k_n}, x_{k_n+1}, \ldots]}\} \leq 2^{-n}\right)$$

is well-defined and $(\delta^e_{\mathrm{dual},=}, [\nu_{\mathbb{N}}]^\omega)$-computable.

3.1.3 The question

Complete orthonormal sequences in separable Hilbert spaces are examples of Schauder bases. The classical *Gram-Schmidt algorithm* (see e.g. [Gro80, Section III.2]) yields that every separable Hilbert space possesses a complete orthonormal sequence and hence possesses a Schauder basis. The following effective version of this fact was noted in [BY06]:

Lemma 3.4. *Let $(X, \|\cdot\|, e)$ be an infinite-dimensional*[1] *computable Banach space such that $(X, \|\cdot\|)$ is a Hilbert space. Then there exists a $[\alpha^e_{\mathrm{Cauchy}}]^\omega$-computable complete orthonormal sequence in X.*

Proof. Apply the Effective Independence Lemma to the sequence e in order to compute a linearly independent complete sequence. Apply the Gram-Schmidt algorithm to this sequence. □

We ask whether this result can be generalized in the following way: Suppose that $(X, \|\cdot\|, e)$ is a computable Banach space that possesses a basis; does there exist an $[\alpha^e_{\mathrm{Cauchy}}]^\omega$-computable basis? The result that we prove in this chapter will be that there exists a computable Banach space that possesses a shrinking basis, but that does not possess any $[\alpha^e_{\mathrm{Cauchy}}]^\omega$-computable basis. The relevance of this question on the one hand stems from the work of Brattka and Dillhage (see previous section); on the other hand, Schauder bases are such an important subject in functional analysis that we think the question is also of independent interest.

3.2 The counter-example

3.2.1 The space of Enflo/Davie

Our construction is built on the following general intuition:

> Suppose that object A has property E, but does not have property E in an effective sense. Then A is "close to not having property E".

The starting point for our construction will hence be a computable Banach space that does not possess a basis. But does there exist such a space at all? Banach, in his classical monograph [Ban32] on linear operators, posed the question whether every separable Banach space possesses a basis. The first counter-example was constructed more than forty years later by Enflo [Enf73]. Enflo in fact even constructed a separable Banach space that lacks the approximation

[1] The finite-dimensional version of this lemma also holds true and is even more simple to prove.

property (cf. Proposition 3.2). A little later, Davie [Dav73] found a simplification of Enflo's example.

Our first task is to check that Davie's space allows a computability structure. We therefore describe Davie's construction: For any $k \in \mathbb{N}$, let G_k be the additive group $\mathbb{Z}/(3 \cdot 2^k)\mathbb{Z}$. For $j = 1, \cdots, 3 \cdot 2^k$, let $\gamma_j^{(k)}$ be the (unique) group homomorphism from G_k into the multiplicative group $\mathbb{C} \setminus \{0\}$ with

$$\gamma_j^{(k)}(1) = \exp\left(2\pi\mathrm{i} \cdot \frac{j}{3 \cdot 2^k}\right).$$

It is shown in [Dav73] (with a probabilistic argument) that there is a constant A_2 such that for every k, the set $\{\gamma_j^{(k)} \ : \ 1 \leq j \leq 3 \cdot 2^k\}$ can be partitioned into two sets $\{\sigma_j^{(k)} \ : \ 1 \leq j \leq 2^k\}$ and $\{\tau_j^{(k)} \ : \ 1 \leq j \leq 2 \cdot 2^k\}$ with

$$(\forall\, g \in G_k) \ \left(\left| 2 \sum_{j=1}^{2^k} \sigma_j^{(k)}(g) - \sum_{j=1}^{2 \cdot 2^k} \tau_j^{(k)}(g) \right| < A_2 (k+1)^{1/2} 2^{k/2} \right).$$

Similarly, it is shown that there is a constant A_3 such that for every $k \geq 1$, there are $\varepsilon_j^{(k)} \in \{-1, 1\}$ ($j = 1, \ldots, 2^k$) with

$$(\forall\, g \in G_k)(\forall\, h \in G_{k-1}) \ \left(\left| \sum_{j=1}^{2^k} \varepsilon_j^{(k)} \frac{\tau_j^{(k-1)}(h)}{\sigma_j^{(k)}(g)} \right| < A_3 (k+1)^{1/2} 2^{k/2} \right).$$

By exhaustive search, $\sigma_j^{(k)}$, $\tau_j^{(k)}$, $\varepsilon_j^{(k)}$ ($1 \leq j \leq 3 \cdot 2^k$) such that the above two inequalities are fulfilled can be found effectively in k. Let G be the disjoint union $\bigcup_{k \in \mathbb{N}} G_k$. Let ν be a computable bijection between the set

$$\{(k, j) \ : \ k, j \in \mathbb{N},\, 1 \leq j \leq 2^k\}$$

and $\mathbb{N}$. We define a mapping e from $\mathbb{N}$ into the linear space of bounded complex functions on G by

$$e(\nu(k,j))(g) := \begin{cases} \tau_j^{(k-1)}(g) & \text{if } k \geq 1 \text{ and } g \in G_{k-1}, \\ \varepsilon_j^{(k)} \sigma_j^{(k)}(g) & \text{if } g \in G_k, \\ 0 & \text{otherwise.} \end{cases}$$

We equip the space of bounded complex functions on G with the $\sup$-norm. In this Banach space, we consider the subspace

$$Z := [e(0), e(1), \ldots].$$

Davie showed that Z lacks AP. Furthermore, it is straightforward to check that $(Z, \|\cdot\|, e)$ is a computable Banach space. We hence have:

Lemma 3.5. *The computable Banach space $(Z, \|\cdot\|, e)$ constructed above lacks AP.* $\square$

3.2.2 Local basis structure

Apart from lacking AP, Davie's space has another property that will be crucial for our argument: A Banach space X is said to have **local basis structure** if there exists a constant C such that for every finite-dimensional subspace V of X, there is a finite-dimensional space W with $V \subseteq W \subseteq X$ and $\mathrm{bc}(W) < C$. This notion was introduced in [Puj75] (under a different name; cf. [Sin81, p. 820]).

For every $n \geq 1$, the Banach space ℓ_∞^n is defined as the linear space $\mathbb{C}^n$ equipped with the $\sup$-norm (cf. [Meg98, Example 1.2.9]). The following criterion for local basis structure is found in [Sza87, Proposition 1.3]:

Proposition 3.6 (Criterion for local basis structure)**.** *Let X be a Banach space such that there exists a constant C such that for every $n \geq 1$, there is a subspace V_n of X and an isomorphism $F_n : V_n \to \ell_\infty^n$ with $\|F_n\|\|F_n^{-1}\| \leq C$. Then X has local basis structure.* □

Corollary 3.7. *The space Z from Lemma 3.5 has local basis structure.*

Proof. For every $n \geq 1$, there is even an isometric isomorphism from a subspace of Z onto ℓ_∞^n: For any $k, j \in \mathbb{N}$, the function $e(\nu(k, j)) : G \to \mathbb{C}$ is supported on $G_{k-1} \cup G_k$. So we can choose $k_1, \dots, k_n$ such that $e(k_1), \dots, e(k_n)$ have pairwise disjoint supports. The norm of Z (just like the norm of ℓ_∞^n) is the $\sup$-norm. So it is obvious that the subspace $[e(k_1), \dots, e(k_n)]$ of Z is isometrically isomorphic to ℓ_∞^n via F_n with

$$F_n(e(k_i)) = (0, \dots, 0, \underbrace{1}_{i\text{-th}}, 0, \dots, 0), \qquad i = 1, \dots, n.$$

(Note that all functions in the range of e have norm 1.) □

A computable Banach space with local basis structure can be effectively "approximated" by finite-dimensional subspaces with uniformly bounded basis constants:

Lemma 3.8. *Let $(X, \|\cdot\|, e)$ be an infinite-dimensional computable Banach space such that X has local basis structure, witnessed by a constant $C \in \mathbb{N}$. There exists an $[\alpha^e_{\mathrm{Cauchy}}]^\omega$-computable linearly independent sequence $(x_i)_i \in X^\omega$ and a strictly increasing computable function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $[x_0, x_1, \dots] = X$ and*

$$(\forall\, n \in \mathbb{N})\ (\mathrm{bc}([x_0, \dots, x_{\sigma(n)}]) < C).$$

Before we turn to the proof, we need an auxiliary lemma:

Lemma 3.9. *Let $(X, \|\cdot\|, e)$ be a computable Banach space.*

(1) *The mapping*

$$(x_1, \dots, x_n) \mapsto \mathrm{bc}((x_1, \dots, x_n))$$

for $(x_1, \dots, x_n) \in \mathrm{IND}_X$ *is* $([\alpha^e_{\mathrm{Cauchy}}]^{<\omega}, \rho_{\mathbb{R}})$*-computable.*

(2) *The mapping*

$$(x_1, \dots, x_n) \mapsto \mathrm{bc}([x_1, \dots, x_n])$$

for $(x_1, \dots, x_n) \in \mathrm{IND}_X$ *is* $([\alpha^e_{\mathrm{Cauchy}}]^{<\omega}, \rho_{\mathbb{R}>})$*-computable.*

Proof. Let an $[\alpha^e_{\mathrm{Cauchy}}]^{<\omega}$-name of some $(x_1, \dots, x_n) \in \mathrm{IND}_X$ be given.

For item (1): By Lemma 2.20, we can $\theta^e_{\mathcal{K}}$-compute the set $B_{[x_1,\dots,x_n]}$ and the coordinate functionals $(f_1, \dots, f_n)$. By Lemma 2.14.2 (and type conversion), we can compute the numbers

$$C_\ell := \sup \left\{ \Big\| \sum_{i=1}^{\ell} x_i f_i(x) \Big\| \ : \ x \in B_{[x_1,\dots,x_n]} \right\}, \qquad \ell = 1 \dots, n.$$

We can now compute $\mathrm{bc}((x_1, \dots, x_n))$ as the maximum of $C_1, \dots, C_n$.

For item (2): As a consequence of item (1), the basis constant of a finite basis depends continuously on the basis' elements. This implies that $\mathrm{bc}([x_1, \dots, x_n])$ is the infimum of the set

$$\left\{ \mathrm{bc}(a_1, \dots, a_n) \ : \ a_1, \dots, a_n \in \mathrm{span}_Q(\{x_1, \dots, x_n\}), \ (a_1, \dots, a_n) \in \mathrm{IND}_X \right\}.$$

By item (1) and Lemma 2.21, we can $[\rho_{\mathbb{R}}]^\omega$-compute a sequence that exhausts this set. We can hence $\rho_{\mathbb{R}>}$-compute its infimum. □

Proof of Lemma 3.8. The function σ and the sequence $(x_i)_i$ are computed recursively as follows: Search for an arbitrary ℓ with $e(\ell) \neq 0$. Put $\sigma(0) = 0$, $x_0 = e(\ell)$. Now suppose that $\sigma(n)$ and $x_0, \dots, x_{\sigma(n)}$ have already been computed. Search for an ℓ such that $x_0, \dots, x_{\sigma(n)}, e(\ell)$ are linearly independent and put $x_{\sigma(n)+1} = e(\ell)$. The search for ℓ in this step is performed using a search scheme as in the proof of the Effective Independence Lemma (Proposition 2.22); this way, we can ensure that eventually

$$[x_0, x_1, \dots] = [e(0), e(1), \dots] = X.$$

(The argument is almost identical to the one from the proof of Proposition 2.22, so we will not go into detail here again.) As X has local basis structure, there must be a number $k \in \mathbb{N}$ and points $a_1, \dots, a_k \in X$ such that

$$(x_0, \dots, x_{\sigma(n)+1}, a_1, \dots, a_k) \in \mathrm{IND}_X$$

and

$$\mathrm{bc}([x_0, \dots, x_{\sigma(n)+1}, a_1, \dots, a_k]) < C.$$

Lemma 3.9.2 yields that suitable $a_1, \dots, a_k$ can then be found in $\mathcal{R}_{\alpha^e}$ (for reasons of continuity) and can furthermore be searched for effectively. So once such $k, a_1, \dots, a_k$ are found, put $\sigma(n+1) = \sigma(n) + 1 + k$ and $x_{\sigma(n)+1+i} = a_i$ for $i = 1, \dots, k$. □

3.2.3 A class of computable spaces with bases

For any sequence $(X_n)_n$ of Banach spaces over $\mathbb{F}$, let $(X_0 \times X_1 \times \cdots)_{c_0}$ be the Banach space of all sequences $(x_n)_n$ with $x_n \in X_n$ for all n and $\lim_{n\to\infty} \|x_n\| = 0$, equipped with the norm

$$\|(x_n)_n\| := \sup_{n\in\mathbb{N}} \|x_n\|.$$

The dual of a space of the form $(X_0 \times X_1 \times \cdots)_{c_0}$ has a simple description in terms of the duals of the spaces X_i:

Lemma 3.10. *Let $(X_n)_n$ be a sequence of Banach spaces over $\mathbb{F}$. Put*

$$X := (X_0 \times X_1 \times \cdots)_{c_0}.$$

If $(f_n)_n$ is a sequence with $f_n \in X_n^*$ for every n and $\sum_{n=0}^{\infty} \|f_n\| < \infty$, then f with

$$f((x_n)_n) = \sum_{n=0}^{\infty} f_n(x_n), \qquad (x_n)_n \in X, \tag{3.2}$$

is a well-defined element of X^* with $\|f\| = \sum_{n=0}^{\infty} \|f_n\|$. Furthermore, every element of X^* is of this form.

Proof. This is a straightforward generalization of [Meg98, Example 1.10.4].

For any functionals $f_0 \in X_0^*, f_1 \in X_1^*, \ldots$ and any $m \in \mathbb{N}$, consider the functional $\widetilde{f}_m \in X^*$ with

$$\widetilde{f}_m((x_n)_n) := \sum_{n=0}^{m} f_n(x_n).$$

Note that

$$|\widetilde{f}_m((x_n)_n)| = \Big|\sum_{n=0}^{m} f_n(x_n)\Big| \le \sum_{n=0}^{m} \|f_n\|\|x_n\| \le \|(x_n)_n\| \sum_{n=0}^{m} \|f_n\|$$

for every $(x_n)_n \in X$, and hence $\|\widetilde{f}_m\| \le \sum_{n=0}^{m} \|f_n\|$. Next, let $\varepsilon > 0$ be arbitrary. For every $0 \le n \le m$, choose $x_n \in B_{X_n}$ such that $|f_n(x_n)| \ge \|f_n\| - \varepsilon/(m+1)$, and choose $\alpha_n \in \mathbb{F}$ with $|\alpha_n| = 1$, $\alpha_n f_n(x_n) \in \mathbb{R}$, and $\alpha_n f_n(x_n) = |f_n(x_n)|$. Then

$$x := (\alpha_0 x_0, \ldots, \alpha_m x_m, 0, \ldots) \in B_X$$

and

$$|\widetilde{f}_m(x)| = \Big|\sum_{n=0}^{m} \alpha_n f_n(x_n)\Big| = \sum_{n=0}^{m} |f_n(x_n)| \ge \sum_{n=0}^{m} \|f_n\| - \varepsilon.$$

As ε was arbitrary, we have $\|\widetilde{f}_m\| \ge \sum_{n=0}^{m} \|f_n\|$. We have shown that $\|\widetilde{f}_m\| = \sum_{n=0}^{m} \|f_n\|$.

Let $(f_n)_n$ be a sequence as in the statement of the lemma. Then

$$\|\widetilde{f}_{m+k} - \widetilde{f}_m\| = \sum_{n=m+1}^{m+k} \|f_n\|,$$

so $(\widetilde{f}_m)_m$ is Cauchy and hence convergent in X^*. So f as in (3.2) is its well-defined limit, and

$$\|f\| = \lim_{m\to\infty} \|\widetilde{f}_m\| = \sum_{n=0}^{\infty} \|f_n\|.$$

To prove the second assertion, let $g \in X^*$ be arbitrary. For every $n \in \mathbb{N}$, let $\mathrm{emb}^{(n)} : X_n \to X$ be the isometric embedding with

$$\mathrm{emb}^{(n)}(x) = (0, \ldots, 0, \underbrace{x}_{\text{index } n}, 0, 0, \ldots,),$$

and define $f_n \in X_n^*$ by $f_n := g \circ \mathrm{emb}^{(n)}$. For every m and every $(x_n)_n \in X$, we have that

$$\widetilde{f}_m((x_n)_n) = g((x_0, \ldots, x_m, 0, \ldots))$$

and hence $\|\widetilde{f}_m\| \leq \|g\|$. So $\sum_{n=0}^{\infty} \|f_n\| < \infty$. By what we have shown above, the $\widetilde{f}_m$ converge to the functional $f \in X^*$ as in (3.2). f and g coincide on all sequences that are eventually zero; these sequences are dense in X, so $f = g$. $\square$

Consider the computable Banach space $(Z, \|\cdot\|, e)$ from Lemma 3.5. Define

$$Y := (Z \times Z \times \cdots)_{c_0}.$$

For every $n \in \mathbb{N}$, let $\mathrm{proj}^{(n)} : Y \to Z$ and $\mathrm{emb}^{(n)} : Z \to Y$ be given by

$$\mathrm{proj}^{(n)}((z_i)_i) := z_n, \qquad (z_i)_i \in Y$$

and

$$\mathrm{emb}^{(n)}(z) := (0, \ldots, 0, \underbrace{z}_{\text{index } n}, 0, 0, \ldots), \qquad z \in Z.$$

Apply Lemma 3.8 to $(Z, \|\cdot\|, e)$; let σ and $(x_i)_i$ be as in the statement of that lemma. For every $n \in \mathbb{N}$, put

$$Z_n := [x_0, \ldots, x_{\sigma(n)}].$$

For every $\tau : \mathbb{N} \to \mathbb{N}$, define

$$Y_\tau := (Z_{\tau(0)} \times Z_{\tau(1)} \times \cdots)_{c_0}.$$

The fact that the Z_n have uniformly bounded basis constants goes into the proof of the following proposition:

Proposition 3.11. *Let $\tau : \mathbb{N} \to \mathbb{N}$ be arbitrary. Then Y_τ has a shrinking basis.*

Proof. By Lemma 3.8, there is a constant C such that every Z_n has a basis $a_{n,0}, \ldots, a_{n,\sigma(n)}$ with basis constant less than C. For every $n \in \mathbb{N}$ and $0 \leq i \leq \sigma(\tau(n))$, put

$$b_{n,i} := \mathrm{emb}^{(n)}(a_{\tau(n),i}).$$

We will show that

$$b_{0,0}, \ldots, b_{0,\sigma(\tau(0))},\ b_{1,0}, \ldots, b_{1,\sigma(\tau(1))},\ \ldots,\ \ldots \tag{3.3}$$

is a shrinking basis of Y_τ.

Let $(z_n)_n \in Y_\tau$ be arbitrary. Suppose that there exists an expansion

$$\alpha_{0,0}b_{0,0} + \cdots + \alpha_{0,\sigma(\tau(0))}b_{0,\sigma(\tau(0))}\ +\ \alpha_{1,0}b_{1,0} + \cdots + \alpha_{1,\sigma(\tau(1))}b_{1,\sigma(\tau(1))}\ +\ \ldots\ +\ \ldots \tag{3.4}$$

of $(z_n)_n$ with respect to the sequence from (3.3). For every n, the continuity of $\mathrm{proj}^{(n)}$ yields

$$z_n = \sum_{i=0}^{\sigma(\tau(n))} \alpha_{n,i} a_{\tau(n),i},$$

so $\alpha_{n,0}, \ldots, \alpha_{n,\sigma(\tau(n))}$ must be the unique coordinates of z_n with respect to the basis

$$a_{\tau(n),0}, \ldots, a_{\tau(n),\sigma(\tau(n))} \tag{3.5}$$

of $Z_{\tau(n)}$. Every element of Y_τ thus has at most one expansion with respect to the sequence from (3.3).

To show that (3.3) is a basis, it remains to show that (3.4) converges to $(z_n)_n \in Y_\tau$ if the $\alpha_{n,i}$ are chosen such that $\alpha_{n,0}, \ldots, \alpha_{n,\sigma(\tau(n))}$ is the expansion of z_n with respect to the basis (3.5) for every n. The partial sums of the series (3.4) have the form

$$\sum_{n=0}^{M-1} \sum_{i=0}^{\sigma(\tau(n))} \alpha_{n,i} b_{n,i} + \sum_{i=0}^{N} \alpha_{M,i} b_{M,i}$$

with $N, M \in \mathbb{N}$, $0 \leq N \leq \sigma(\tau(M))$. We have the following estimate for the distance to $(z_n)_n$:

$$\begin{aligned}
&\big\|(z_n)_n - \big(\sum_{n=0}^{M-1}\sum_{i=0}^{\sigma(\tau(n))} \alpha_{n,i}b_{n,i} + \sum_{i=0}^{N}\alpha_{M,i}b_{M,i}\big)\big\| \\
&= \big\|\sum_{n=0}^{\infty}\mathrm{emb}^{(n)}(z_n) - \sum_{n=0}^{M-1}\sum_{i=0}^{\sigma(\tau(n))} \alpha_{n,i}\mathrm{emb}^{(n)}(a_{\tau(n),i}) - \sum_{i=0}^{N}\alpha_{M,i}\mathrm{emb}^{(M)}(a_{\tau(M),i})\big\| \\
&= \big\|\sum_{n=0}^{\infty}\mathrm{emb}^{(n)}(z_n) - \sum_{n=0}^{M-1}\mathrm{emb}^{(n)}\big(\underbrace{\sum_{i=0}^{\sigma(\tau(n))}\alpha_{n,i}a_{\tau(n),i}}_{=z_n}\big) - \mathrm{emb}^{(M)}\big(\sum_{i=0}^{N}\alpha_{M,i}a_{\tau(M),i}\big)\big\| \\
&= \big\|\sum_{n=M}^{\infty}\mathrm{emb}^{(n)}(z_n) - \mathrm{emb}^{(M)}\big(\sum_{i=0}^{N}\alpha_{M,i}a_{\tau(M),i}\big)\big\| \\
&= \max\big(\sup_{n>M}\|z_n\|,\ \big\|z_M - \sum_{i=0}^{N}\alpha_{M,i}a_{\tau(M),i}\big\|\big) \\
&\leq \max\big(\sup_{n>M}\|z_n\|,\ \|z_M\| + \big\|\sum_{i=0}^{N}\alpha_{M,i}a_{\tau(M),i}\big\|\big) \\
&\leq \max\big(\sup_{n>M}\|z_n\|,\ \|z_M\| + C\|z_M\|\big).
\end{aligned}$$

In view of the fact that $\|z_M\| \to 0$ as $M \to \infty$, this estimate yields the convergence of (3.4) to $(z_n)_n$.

It remains to show that the basis (3.3) is shrinking. Let $f \in (Y_\tau)^*$ be arbitrary. f has the form

$$f((z_n)_n) = \sum_{n=0}^{\infty} f_n(z_n)$$

with certain $f_n \in (Z_{\tau(n)})^*$ ($n \in \mathbb{N}$), and $\|f\| = \sum_{n=0}^{\infty}\|f_n\|$ (see Lemma 3.10). For every ℓ, let B_ℓ be the closed unit ball in the subspace

$$\begin{aligned}
&[b_{\ell,0}, \ldots, b_{\ell,\sigma(\tau(\ell))},\ b_{\ell+1,0}, \ldots, b_{\ell+1,\sigma(\tau(\ell+1))},\ \ldots,\ \ldots] \\
&\qquad\qquad = (\underbrace{\{0\} \times \cdots \times \{0\}}_{\ell\text{-times}} \times Z_{\tau(\ell)} \times Z_{\tau(\ell+1)} \times \cdots)_{c_0}
\end{aligned}$$

of Y. Then (again by Lemma 3.10)

$$\sup\{|f(y)| \ :\ y \in B_\ell\} = \sum_{n=\ell}^{\infty}\|f_n\|,$$

so

$$\lim_{\ell\to\infty}\sup\{|f(y)| \ :\ y \in B_\ell\} = 0.$$

This completes the proof. $\square$

The following lemma and its corollary will be useful in the next section:

Lemma 3.12 (AP is inherited by complemented subspaces)**.** *Let X be a Banach space that has AP, and let V be a closed subspace such that there is an $F \in B(X)$ with* $\mathrm{range}(F) = V$ *and $F(v) = v$ for every $v \in V$. Then V has AP.*

Proof. The claim is trivial if $V = \{0\}$. So suppose otherwise. Then necessarily $\|F\| \geq 1$, in particular $\|F\| > 0$. Let K be compact in V and $\varepsilon > 0$ arbitrary. K is also compact in X. As X has AP, there is $G \in \mathrm{FR}(X, X)$ such that

$$\sup_{x \in K} \|G(x) - x\| \leq \varepsilon \|F\|^{-1}.$$

Put $G' := F \circ G|_V$. Then $G' \in \mathrm{FR}(V)$. Furthermore

$$\sup_{x \in K} \|G'(x) - x\| = \sup_{x \in K} \|F(G(x)) - F(x)\| \leq \|F\| \sup_{x \in K} \|G(x) - x\| \leq \varepsilon.$$

□

Corollary 3.13. *Let $(y_i)_i \in Y^\omega$ be a basic sequence. Then*

$$\mathrm{emb}^{(n)}(Z) \not\subseteq [y_0, y_1, \ldots].$$

for every $n \in \mathbb{N}$.

Proof. Let us assume that $\mathrm{emb}^{(n)}(Z) \subseteq [y_0, y_1, \ldots]$ for some n. The space $X := [y_0, y_1, \ldots]$ has a basis and hence has AP. The mapping

$$F := \mathrm{emb}^{(n)} \circ \mathrm{proj}^{(n)}|_X$$

is in $B(X, X)$ with $\mathrm{range}(F) = \mathrm{emb}^{(n)}(Z)$ and $F(v) = v$ for every $v \in \mathrm{emb}^{(n)}(Z)$. The previous lemma yields that $\mathrm{emb}^{(n)}(Z)$ has AP. $\mathrm{emb}^{(n)}(Z)$, however, is isometrically isomorphic to Z, which lacks AP. Contradiction! □

Let us finally equip Y and the Y_τ with computability structures: It is straightforward to verify that $(Y, \|\cdot\|, h)$ with

$$h(\langle n, i \rangle) := \mathrm{emb}^{(n)}(e(i)), \qquad n, i \in \mathbb{N},$$

is a computable Banach space. Recall that a function $\tau : \mathbb{N} \to \mathbb{N}$ is **lower semicomputable** if there is a c.e. set $N \subseteq \mathbb{N}$ with

$$\tau(n) = \sup\{k \in \mathbb{N} \ : \ \langle n, k \rangle \in N\}, \qquad n \in \mathbb{N}.$$

If τ is lower semicomputable, it is easy to see that there is an $[\alpha^h_{\mathrm{Cauchy}}]^\omega$-computable enumeration h_τ of the set[2]

$$\{\mathrm{emb}^{(n)}(x_i) \ : \ n, i \in \mathbb{N}, \, 0 \leq i \leq \sigma(\tau(n))\};$$

the span of this set is dense in Y_τ. We learn the following from [Bra01, Proposition 3.10]:

[2]Recall that σ and the x_i were defined on page 59.

Lemma 3.14 (Y_τ as a subspaces of Y)**.** *Let $\tau : \mathbb{N} \to \mathbb{N}$ be lower semicomputable. Then $(Y_\tau, \|\cdot\|, h_\tau)$ is a computable Banach space. The injection $Y_\tau \hookrightarrow Y$ is $(\alpha^{h_\tau}_{\text{Cauchy}}, \alpha^{h}_{\text{Cauchy}})$-computable.* □

3.2.4 The diagonalization construction

In this section, we will prove the main result of this chapter:

Theorem 3.15. *There exists a lower-semicomputable $\tau : \mathbb{N} \to \mathbb{N}$ such that the computable Banach space $(Y_\tau, \|\cdot\|, h_\tau)$ as defined above possesses a basis, but does not possess any $[\alpha^{h_\tau}_{\text{Cauchy}}]^\omega$-computable basis.*

In view of the results of the previous subsection, it remains to construct a lower-semicomputable τ such that $(Y_\tau, \|\cdot\|, h_\tau)$ does not possess any $[\alpha^{h_\tau}_{\text{Cauchy}}]^\omega$-computable basis. By Lemma 3.14, every $[\alpha^{h_\tau}_{\text{Cauchy}}]^\omega$-computable sequence is $[\alpha^{h}_{\text{Cauchy}}]^\omega$-computable. So it is sufficient to compute τ such that every $[\alpha^{h}_{\text{Cauchy}}]^\omega$-computable sequence $(y_i)_i \in Y^\omega$ has one of the following two properties:

- $(y_i)_i$ is not basic.
- $Y_\tau \not\subseteq [y_0, y_1, \ldots]$.

We will proceed by diagonalization over all $[\alpha^{h}_{\text{Cauchy}}]^\omega$-computable sequences. Let us first note the following fact which follows immediately from the definition of the Cauchy representation: For every $[\alpha^{h}_{\text{Cauchy}}]^\omega$-computable sequence $(y_i)_i$, there is a total computable function $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $(\alpha^h(\psi(i,k)))_k$ converges rapidly to y_i for every i. It is well-known that there exists a universal partial computable $\Psi :\subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$; that means, for every partial computable $\psi :\subseteq \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, there is an $n \in \mathbb{N}$ such that

$$(n, i, k) \in \mathrm{dom}(\Psi) \Leftrightarrow (i, k) \in \mathrm{dom}(\psi)$$

and

$$(i, k) \in \mathrm{dom}(\psi) \Rightarrow \Psi(n, i, k) = \psi(i, k)$$

for all $i, k \in \mathbb{N}$. In this case, n is called a **Gödel number** of ψ. Let ψ_n be the partial computable function with Gödel number n. Denote by TOT the set of all n such that ψ_n is total. Denote by SEQ the set of all n such that ψ_n corresponds to an $[\alpha^{h}_{\text{Cauchy}}]^\omega$-computable sequence in the way described above.

Lemma 3.16. *There is a computably enumerable set $M \subseteq \mathbb{N}$ with*

$$\mathrm{TOT} \setminus \mathrm{SEQ} \subseteq M \subseteq \mathbb{N} \setminus \mathrm{SEQ}.$$

Proof. M shall be defined as the set of all n with

$$(\exists\, i \in \mathbb{N})(\exists\, k \in \mathbb{N})(\exists\, j \in \mathbb{N})$$
$$\big(j > k, \quad (i,k) \in \mathrm{dom}(\psi_n), \quad (i,j) \in \mathrm{dom}(\psi_n), \quad \text{and} \quad \|\alpha^h(\psi_n(i,k)) - \alpha^h(\psi_n(i,j))\| > 2^{-k}\big),$$

which is easily seen to be computably enumerable. The claimed inclusions follow directly from the definition of rapid convergence. □

We now define a lower-semicomputable $\tau : \mathbb{N} \to \mathbb{N}$ by giving an algorithm that enumerates a set $L\langle n,k\rangle_{**} \subseteq \mathbb{N}$ with

$$\tau\langle n,k\rangle_{**} = \max(L\langle n,k\rangle_{**} \cup \{0\}) \tag{3.6}$$

for any given $\langle n,k\rangle_{**} \in \mathbb{N}$.

We begin the description of the algorithm: Let $\langle n,k\rangle_{**}$ be given. The procedure consists of four parallel processes **A**, **B**, **C**, **D**. The set $L\langle n,k\rangle_{**}$ is defined to be the intersection of the sets put out by processes **A** and **D**.

Process A runs a loop over $\ell = 0, 1, 2, \ldots$. In the body of the loop, ψ_n is called with input (i,j) chosen such that $\ell = \langle i,j\rangle_{**}$. If this call returns, ℓ is put out and the loop continues.

Process B runs a semidecision procedure for "$n \in M$", where M is the set from Lemma 3.16. As soon as "$n \in M$" (if ever), the process immediately terminates itself as well as **A**, **C**, and **D**.

We will only define the behaviour of processes **C**, **D** for $n \in \mathrm{SEQ}$. For $n \notin \mathrm{SEQ}$, the behaviour of **C** and **D** shall be undefined. So let $(y_i)_i \in Y^\omega$ be the computable sequence corresponding to n.

Process C performs an exhaustive search for $\ell, m \in \mathbb{N}$, $\ell \le m$, and $\alpha_0, \ldots, \alpha_m \in \mathbb{Q}[\mathrm{i}]$ with

$$\Big\|\sum_{i=0}^{\ell} \alpha_i y_i\Big\| > k \Big\|\sum_{i=0}^{m} \alpha_i y_i\Big\|.$$

Once such numbers are found, the process immediately terminates itself as well as **A**, **B**, and **D**.

Process D performs a loop over $\ell = 0, 1, \ldots$. In the body of the loop, first ℓ is put out, then an exhaustive search for elements $\widetilde{x}_0^{(\ell)}, \ldots, \widetilde{x}_{\sigma(\ell)}^{(\ell)}$ of $\mathrm{span}_{\mathbb{Q}[\mathrm{i}]}(\{y_i \;:\; i \in \mathbb{N}\})$ with

$$(\forall\, 0 \le i \le \sigma(\ell))\ \big(\|\mathrm{emb}^{(\langle n,k\rangle_{**})}(x_i) - \widetilde{x}_i^{(\ell)}\| < 2^{-\ell}\big)$$

is performed.[3] In case such elements are found, the loop continues.

This completes the description of the algorithm.

[3]Recall that σ and the x_i were defined on page 59.

$\tau : \mathbb{N} \to \mathbb{N}$ is well-defined by (3.6) if, and only if, $L\langle n,k\rangle_{**}$ is finite for all n, k. So we have to make sure that the output of either **A** or **D** is finite:
Case 1: $n \notin \mathrm{TOT}$. Then **A** will sooner or later call the function ψ_n with an argument from outside $\mathrm{dom}(\psi_n)$. This call will not return, so the process will "hang" and not produce any more output.
Case 2: $n \in \mathrm{TOT} \setminus \mathrm{SEQ}$. **B** will sooner or later detect that $n \in M$, so all processes are terminated after finite time.
Case 3: Otherwise. Then $n \in \mathrm{SEQ}$. Let $(y_i)_i$ be the corresponding sequence.
Case 3a: The nonzero elements of $(y_i)_i$ form a basic sequence. We show that the loop in process **D** will only be iterated a finite number of times: Suppose the contrary. Then all $\mathrm{emb}^{(\langle n,k\rangle_{**})}(x_i)$, $i \in \mathbb{N}$, can be approximated arbitrarily well by elements from $\mathrm{span}_Q(\{y_i \ : \ i \in \mathbb{N}\})$. This implies

$$\{\mathrm{emb}^{(\langle n,k\rangle_{**})}(x_i) \ : \ i \in \mathbb{N}\} \subseteq [y_0, y_1, \ldots],$$

and thus

$$\begin{aligned}\mathrm{emb}^{(\langle n,k\rangle_{**})}(Z) = \mathrm{emb}^{(\langle n,k\rangle_{**})}([x_0, x_1, \ldots]) &= [\mathrm{emb}^{(\langle n,k\rangle_{**})}(x_0), \mathrm{emb}^{(\langle n,k\rangle_{**})}(x_1), \ldots] \\ &\subseteq [y_0, y_1, \ldots].\end{aligned}$$

This contradicts Corollary 3.13.
Case 3b: Otherwise. The nonzero elements of $(y_i)_i$ do not form a basic sequence. Proposition 3.1 ensures that the exhaustive search performed by process **C** will succeed, so all processes will sooner or later be terminated.

It remains to show that $Y_\tau \not\subseteq [y_0, y_1, \ldots]$ for any computable basic sequence $(y_i)_i \in Y^\omega$: Let n be a Gödel number of $(y_i)_i$. Choose $k \in \mathbb{N}$ greater than the basis constant of $(y_i)_i$. As

$$\mathrm{emb}^{(\langle n,k\rangle_{**})}(Z_{\tau(\langle n,k\rangle_{**})}) \subseteq Y_\tau,$$

it is sufficient to show

$$\mathrm{emb}^{(\langle n,k\rangle_{**})}(Z_{\tau(\langle n,k\rangle_{**})}) \not\subseteq [y_0, y_1, \ldots].$$

This is fulfilled if, and only if,

$$\{\mathrm{emb}^{(\langle n,k\rangle_{**})}(x_i) \ : \ 0 \le i \le \sigma(\tau(\langle n,k\rangle_{**}))\} \not\subseteq [y_0, y_1, \ldots]. \tag{3.7}$$

As $n \in \mathrm{SEQ}$, we have $n \notin M$, so process **B** will not terminate the other processes. As $k > \mathrm{bc}((y_i)_i)$, the exhaustive search performed by process **C** will not succeed, so **C** will not terminate the other processes, either. Process **A** will enumerate the entire set $\mathbb{N}$. Together, this implies that $L\langle n,k\rangle_{**}$ is equal to the output of process **D**. Consider the final iteration of the loop in process **D**, that means $\ell = \tau(\langle n,k\rangle_{**})$. The exhaustive search in the body of the loop does not succeed (otherwise, this were not the final iteration). This directly implies (3.7). The proof is complete.

3.3 Further directions

In the future, a more complete understanding of the problem studied in this chapter could be achieved using the notion of computable reducibility described in Section 2.6. In terms of that reducibility, the uncomputability of the multi-valued mapping that maps a Banach space with a basis to one of its bases could be characterized. A suitable representation of Banach spaces was recently introduced in [GM08].

Another direction would be to investigate whether it helps to restrict the problem of finding a basis to spaces that possess bases with special properties. One could ask, for example: *Does every computable Banach space that possesses a monotone basis necessarily posses a computable (monotone) basis?* Here "monotone" may be replaced by one of the many other special properties of bases studied in the literature (see [Sin70, Sin81]). Our example, however, already shows that it does not help to assume the existence of a *shrinking* basis.

Chapter 4

Computability Properties of the Generalized Inverse

4.1 The generalized inverse

Suppose that X and Y are linear spaces over $\mathbb{F}$ and $T : X \to Y$ is linear. In general, the inverse T^{-1} suffers from two shortcomings: Its domain is only a proper subset of Y, and it is not single-valued. If X and Y are Hilbert spaces, one can do something about these shortcomings by replacing the usual inverse by the **generalized inverse** (see e.g. [Gro77, Gro80, EHN96]). The generalized inverse $T^\dagger$ of T is defined in the following way: Let $P : Y \to Y$ be the orthogonal projection operator onto $\overline{\mathrm{range}(T)}$. $y \in Y$ is in the domain of $T^\dagger$ if, and only if, $Py \in \mathrm{range}(T)$. This way, the domain of $T^\dagger$ is not merely $\mathrm{range}(T)$, but $\mathrm{range}(T) \oplus \mathrm{range}(T)^\perp$. The set of preimages of Py may consist of more than one point. This set, however, is closed and convex, and so it possesses a unique minimum-norm element (see [Gro80, Theorem 3.1.2]). $T^\dagger(y)$ is defined to be that element. Note that $T^\dagger = T^{-1}$ if, and only if, T is injective and $\overline{\mathrm{range}(T)} = Y$.

We have already mentioned in the introduction that many operators T that come up in applications are compact and that their inverses (if they are well-defined) are typically discontinuous. For generalized inverses this is still true because $T^\dagger$ is continuous if, and only if, $\mathrm{range}(T)$ is closed (see [Gro77, Theorem 3.1.2] or [EHN96, Proposition 2.4]), but the range of a compact T is closed if, and only if, T has finite rank (see e.g. [Meg98, Proposition 3.4.6]).

4.2 How uncomputable is $(T, y) \mapsto T^\dagger y$?

Consider the canonical infinite-dimensional computable Hilbert space $(\ell_2, \|\cdot\|, e)$ over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ (see Subsection 2.4.4). The aim of this section is a partial characterization of the uncomputability of the mapping

$$\mathrm{GI} :\subseteq B(\ell_2) \times \ell_2 \to \ell_2$$

with

$$\mathrm{dom}(\mathrm{GI}) := \{(T, y) \ : \ y \in \mathrm{dom}(T^\dagger)\}$$

and

$$\mathrm{GI}(T, y) := T^\dagger y.$$

We will find the following:

Theorem 4.1. $(\mathrm{GI}; [\Delta, \alpha^e_{\text{Cauchy}}], \alpha^e_{\text{Cauchy}})$ *is* Σ^0_2*-complete.*

The proof of Theorem 4.1 will use the following two results:

Lemma 4.2 (A single Σ^0_2-hard inverse)**.** *There exists a* $\delta^{e,e}_{\text{ev}}$*-computable self-adjoint* $T \in B(\ell_2)$ *with* $T^\dagger = T^{-1}$ *such that* $(T^\dagger; \alpha^e_{\text{Cauchy}}, \alpha^e_{\text{Cauchy}})$ *is* Σ^0_2*-hard.*

Proof. Consider the mapping $T \in B(\ell_2)$ with $Te(i) := 2^{-i}e(i)$, $i \in \mathbb{N}$. T is obviously $\delta^{e,e}_{\text{ev}}$-computable and self-adjoint. It is also obvious that T is injective and its range is dense in ℓ_2. This yields that $T^\dagger$ is equal to T^{-1}. In order to prove the Σ^0_2-hardness, it remains to verify that $T^\dagger$ fulfills the conditions of Theorem 2.31: T is compact (cf. [Meg98, Example 3.4.5]) and its range is infinite dimensional, so $T^\dagger$ is unbounded (see Section 4.1). $\mathrm{graph}(T^\dagger)$ is closed by [EHN96, Proposition 2.4]. Choose $(y_i)_i := (Te(i))_i$. Then $(y_i)_i$ is computable and $[y_0, y_1, \ldots] = \ell_2$. The sequence $(T^\dagger y_i)_i$ (which is simply e) is computable, too. □

Proposition 4.3 (Approximations to generalized inverses)**.** *There is a* $(\Delta, [\delta^{e,e}_{\text{ev}}]^\omega)$*-computable mapping*

$$\mathrm{TYKH} : B(\ell_2) \to (B(\ell_2))^\omega$$

such that $(F_i)_i = \mathrm{TYKH}(T)$ *implies that each* F_i *is self-adjoint and,*

(1) $(F_i T^* y)_i$ *converges to* $T^\dagger y$ *for every* $y \in \mathrm{dom}(T^\dagger)$*, and*

(2) $\|T^\dagger y - F_i T^* a_i\|^2 \leq \|T^\dagger y\|^2 - \|F_i T^* y\|^2$ *for every* $y \in \mathrm{dom}(T^\dagger)$*,* $i \in \mathbb{N}$*.*

Before we prove Proposition 4.3, we use it do derive Theorem 4.1:

Proof of Theorem 4.1. Let T be the mapping from Lemma 4.2. Note that for every $y \in \mathrm{dom}(T^\dagger)$, we have

$$T^\dagger y = \mathrm{POST}(y, \mathrm{GI}(\mathrm{PRE}(y))),$$

where PRE is the $(\alpha^e_{\text{Cauchy}}, [\Delta, \alpha^e_{\text{Cauchy}}])$-computable mapping $y \mapsto (T, y)$ and POST is the $([\alpha^e_{\text{Cauchy}}, \alpha^e_{\text{Cauchy}}], \alpha^e_{\text{Cauchy}})$-computable mapping $(x, y) \mapsto y$. Thus

$$(T^\dagger; \alpha^e_{\text{Cauchy}}, \alpha^e_{\text{Cauchy}}) \leq_c (\mathrm{GI}; [\Delta, \alpha^e_{\text{Cauchy}}], \alpha^e_{\text{Cauchy}}).$$

By transitivity, we have that $(\mathrm{GI}; [\Delta, \alpha^e_{\mathrm{Cauchy}}], \alpha^e_{\mathrm{Cauchy}})$ is Σ^0_2-hard.

It remains to show that $(\mathrm{GI}; [\Delta, \alpha^e_{\mathrm{Cauchy}}], \alpha^e_{\mathrm{Cauchy}})$ is Σ^0_2-computable. To this end, it is sufficient to prove it reducible to $(\mathrm{LIM}; [\alpha^e_{\mathrm{Cauchy}}]^\omega, \alpha^e_{\mathrm{Cauchy}})$ (see Proposition 2.32). So it is sufficient to demonstrate how to compute, for any given $(T, y) \in \mathrm{dom}(\mathrm{GI})$, a sequence that converges to $T^\dagger y$. Apply TYKH from Proposition 4.3 to T; let $(F_i)_i$ be the result. Then we can compute the sequence $(F_i T^* y)_i$, which has the desired property. □

Let us also note a corollary to Proposition 4.3, which might be useful in some situations:

Corollary 4.4. *The mapping*

$$\mathrm{GI}_1 :\subseteq B(\ell_2) \times \ell_2 \times \mathbb{R} \to \ell_2$$

with

$$\mathrm{dom}(\mathrm{GI}_1) = \{(T, y, c) \; : \; y \in \mathrm{dom}(T^\dagger), \; c = \|T^\dagger y\|\}$$

and

$$\mathrm{GI}_1(T, y, c) = T^\dagger y$$

is $([\Delta, \alpha^e_{\mathrm{Cauchy}}, \rho_{\mathbb{R}>}], \alpha^e_{\mathrm{Cauchy}})$*-computable.*

Proof. Let input (T, y, c) be given. It is sufficient to demonstrate how to compute a 2^{-k}-approximation to $T^\dagger y$ for any given k. Applying TYKH from Proposition 4.3 to T yields a $[\delta^{e,e}_{\mathrm{ev}}]^\omega$-name of a sequence $(F_i)_i$ such that $(F_i T^* y)_i$ converges to $T^\dagger y$. In view of the fact that we are given a $\rho_{\mathbb{R}>}$-name of $c = \|T^\dagger y\|$, we can $[\rho_{\mathbb{R}>}]^\omega$-compute the sequence $(b_i)_i$ with

$$b_i := \sqrt{\|T^\dagger y\|^2 - \|F_i T^* y\|^2}, \qquad i \in \mathbb{N}.$$

This sequence converges to zero, and we can effectively find an i_0 such that $b_{i_0} < 2^{-k}$. Property (2) of TYKH yields that

$$\|T^\dagger - F_{i_0} T^* y\| < 2^{-k}.$$

□

It remains to prove Proposition 4.3. The proof is based on the following result from the theory of *Tykhonov regularization* (see [EHN96, Theorem 4.1 and p. 117]):

Proposition 4.5. *Let I be the identity operator on ℓ_2. Suppose $T \in B(\ell_2)$ and $y \in \ell_2$. For every $t > 0$, the operator*

$$L_t := tI + T^* T$$

is invertible. Put

$$x_t := L_t^{-1} T^* y.$$

Then $(x_t)_{t>0}$ converges for $t \to 0$ if, and only if, $y \in \mathrm{dom}(T^\dagger)$. In this case

$$\lim_{t \to 0} x_t = T^\dagger y.$$

□

In order to prove an amendment to this result, we will use *operator calculus* (see e.g. [Gro80, Chapter V]):

Proposition 4.6 (Operator calculus). *Let $T \in B(\ell_2)$ be self-adjoint, let a, b be numbers such that the spectrum of T is contained in $(a; b]$. There is a continuous homomorphism $f \mapsto f(T)$ from the Banach algebra[1] $C[a; b]$ of real continuous functions on $[a; b]$ into the Banach algebra $B(\ell_2)$ such that*

(1) $f(T) = T$ if $f = \mathrm{id}_{[a;b]}$.

(2) $f(T)$ is non-negative[2] if $f \geq 0$.

(3) $f(T)$ is self-adjoint.

□

Lemma 4.7. *Let T, y, L_t, x_t be as in Proposition 4.5. Then*

$$\|T^\dagger y - x_t\|^2 \leq \|T^\dagger y\|^2 - \|x_t\|^2.$$

Proof. Suppose $0 < s < t$. If we show

$$\|x_s - x_t\|^2 \leq \|x_s\|^2 - \|x_t\|^2, \tag{4.1}$$

the claim follows by letting $s \to 0$. Choose $a \in (-s; 0)$. The spectrum of T^*T is contained in $[0; \|T\|^2]$ (and thus in $(a, \|T\|^2]$), as follows easily from [Gro80, Theorem 3.3.8]. Define $f_t, f_s \in C[a; \|T\|^2]$ by

$$f_t(\lambda) := \frac{1}{t + \lambda}, \qquad f_s(\lambda) := \frac{1}{s + \lambda}.$$

Using operator calculus, we can write $L_t^{-1} = f_t(T^*T)$ and $L_s^{-1} = f_s(T^*T)$. Note that $f_s > f_t > 0$, and thus $f_s f_t - f_t^2 > 0$; so $(f_s f_t - f_t^2)(T^*T)$ is non-negative by item (2) of Proposition 4.6. This implies

$$\begin{aligned} 0 &\leq \langle (f_s f_t - f_t^2)(T^*T)g \mid g\rangle = \langle f_s(T^*T) f_t(T^*T)g \mid g\rangle - \langle f_t(T^*T) f_t(T^*T)g \mid g\rangle \\ &= \langle f_t(T^*T)g \mid f_s(T^*T)g\rangle - \|f_t(T^*T)g\|^2 \end{aligned}$$

and hence

$$\begin{aligned} \|f_s(T^*T)g - f_t(T^*T)g\|^2 &= \|f_s(T^*T)g\|^2 - 2\langle f_t(T^*T)g \mid f_s(T^*T)g\rangle + \|f_t(T^*T)g\|^2 \\ &\leq \|f_s(T^*T)g\|^2 - \|f_t(T^*T)g\|^2 \end{aligned}$$

for any $g \in \ell_2$. This estimate yields (4.1) if one chooses $g = T^*y$. □

[1] See [Meg98, Definition 3.3.1, Example 3.3.3, Example 3.3.7].
[2] Recall that an operator $T \in B(\ell_2)$ is non-negative if $\langle Tx \mid x\rangle \geq 0$ for all $x \in \ell_2$.

We are finally ready for the

Proof of Proposition 4.3. It follows from Proposition 4.5 and Lemma 4.7 that $\mathrm{TYKH}(T) := (L_{2^{-i}}^{-1})_i$ with

$$L_t := tI + T^*T, \qquad t > 0,$$

has the required properties. So it is sufficient to demonstrate how to $\delta_{\mathrm{ev}}^{e,e}$-compute L_t^{-1} from any given $\rho_{\mathbb{R}}$-name of t. This can be done using *effective operator calculus* (see e.g. [Dil08]) or, more elementarily, as follows: We can compute the sequence $(L_t e(n))_n$, which is dense in ℓ_2, and we can trivially compute the sequence $(L_t^{-1} L_t e(n))_n$. A straight-forward calculation shows that $\|L_t x\| \geq t\|x\|$ for any $x \in \ell_2$; this implies $\|L_t^{-1}\| \leq 1/t$. We thus have all we need to compute a $\delta_{\mathrm{seq},\geq}^{e,e}$-name of L_t^{-1}, which can be converted into a $\delta_{\mathrm{ev}}^{e,e}$-name by Lemma 2.23. $\square$

4.3 Further directions

The main motivation for this chapter was to provide a number of tools to be used in Chapter 7, not to study the (un)computability of generalized inverses in detail. A number of directions remain to be explored:

- In our characterization of the uncomputability of GI, we used the Δ-representation of $B(\ell_2)$, which combines $\delta_{\mathrm{ev}}^{e,e}$-information on an operator and its adjoint. It would be desirable to also characterize the uncomputability of $(\mathrm{GI}; [\delta_{\mathrm{ev}}^{e,e}, \alpha_{\mathrm{Cauchy}}^{e}], \alpha_{\mathrm{Cauchy}}^{e})$; we conjecture that this problem is Σ_3^0-complete.
- One should characterize the uncomputability of the restriction of GI to operators whose generalized inverses are continuous. (For the usual inverse, compare [Bra].)
- One should characterize the uncomputability of generalized inverses of operators on finite-dimensional Hilbert spaces. (For the usual inverse, compare [ZB04].)

Chapter 5

Probabilistic Computability on Represented Spaces

5.1 Preliminaries from measure theory

5.1.1 Completion of a measure space

Let $(X, \mathcal{S}, \nu)$ be a measure space. A set $N \subseteq X$ is called **ν-null** if there is a set $B \in \mathcal{S}$ with $\nu(B) = 0$ and $N \subseteq B$. A property $P \subseteq X$ is said to hold **ν-almost everywhere (ν-a.e.)** if $X \setminus P$ is ν-null. The σ-algebra $\mathcal{S}_\nu$ generated by $\mathcal{S}$ and all ν-null sets is called the **completion** of $\mathcal{S}$ with respect to ν. $\mathcal{S}_\nu$ contains exactly the sets of the form $A \cup N$ with $A \in \mathcal{S}$ and N ν-null. We call the elements of $\mathcal{S}_\nu$ the **ν-measurable** sets. The measure ν extends to a measure $\overline{\nu}$ on $\mathcal{S}_\nu$ by putting $\overline{\nu}(A \cup N) = \nu(A)$. A measure space that is identical to its completion is called **complete**.

Lemma 5.1. *Let $(X, \mathcal{S}, \nu)$ be a complete measure space and $(Y, \mathcal{S}')$ a measurable space. Let $f : X \to Y$ be a mapping such that $f|_{X \setminus N}$ is $(\mathcal{S} \cap (X \setminus N), \mathcal{S}')$-measurable for some ν-null set N. Then f is $(\mathcal{S}, \mathcal{S}')$-measurable.*

Proof. Let $A \in \mathcal{S}'$ be arbitrary. We need to show $f^{-1}(A) \in \mathcal{S}$. By assumption, $f^{-1}|_{X \setminus N}(A) \in \mathcal{S} \cap (X \setminus N)$. So there is a set $B \in \mathcal{S}$ with $f^{-1}|_{X \setminus N}(A) = B \cap (X \setminus N)$. We have

$$f^{-1}(A) = (f^{-1}(A) \cap N) \ \cup \ f^{-1}|_{X \setminus N}(A) = (f^{-1}(A) \cap N) \ \cup \ (B \cap (X \setminus N)).$$

The completeness of the measure space yields that the ν-null sets N and $f^{-1}(A) \cap N$ are in $\mathcal{S}$. So $f^{-1}(A) \in \mathcal{S}$. $\square$

If $(X, \mathcal{S}, \nu)$ is a measure space and the function $f : X \to [-\infty, \infty]$ is $(\mathcal{S}_\nu, \mathcal{B}([-\infty, \infty]))$-measurable[1], we call f a **ν-measurable** function. If f is μ-measurable, we will simply write

$$\int f \, d\nu := \int f \, d\overline{\nu}.$$

Similarly, if $(Y, \mathcal{S}')$ is another measurable space and f is an $(\mathcal{S}_\nu, \mathcal{S}')$-measurable mapping, we write

$$\nu \circ f^{-1} := \overline{\nu} \circ f^{-1}$$

for the image of $\overline{\nu}$ under f.[2]

5.1.2 Outer measures

An **outer measure** on a set X is a set function $\mu^* : 2^X \to [0; \infty]$ such that

$$\mu^*(\emptyset) = 0, \qquad A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B), \qquad \mu^*\big(\bigcup_n A_n\big) \leq \sum_{n=0}^{\infty} \mu^*(A_n)$$

for any $A, B, A_0, A_1, \ldots \in 2^X$. A set $A \subseteq X$ is called **μ^*-measurable** if

$$(\forall\, E \subseteq X)\ (\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)).$$

The μ^*-measurable sets form a σ-algebra MEAS_{μ^*}. Restricting μ^* to MEAS_{μ^*} yields a complete measure space (see e.g. [Coh80, Theorem 1.3.4 and Proposition 1.3.3]).

Let $(X, \mathcal{S}, \nu)$ be a measure space. The measure ν induces an outer measure ν^* via

$$\nu^*(A) := \inf\{\nu(B) \ : \ B \in \mathcal{S},\ A \subseteq B\}$$

(see e.g. [Coh80, Proposition 1.5.4]).

If ν is σ-finite[3], it turns out that $\mathrm{MEAS}_{\nu^*} = \mathcal{S}_\nu$, and that $\overline{\nu}$ and ν^* coincide on this σ-algebra (see e.g. [Coh80, Exercise 1.5.9]).

It is known that not every outer measure μ^* is of the form ν^* for some measure ν.

The following two results are actually well-known but usually not stated for outer measures. We will use the second one in the proof of Proposition 5.12.

[1] Here $[-\infty, \infty]$ is regarded as a compactification of $\mathbb{R}$. See [Kal02, p. 5].
[2] These are the conventions applied in the reference [Bog98], that we will heavily work with in Chapter 6.
[3] Recall that ν is σ-finite if there is a sequence $(A_n)_n \in \mathcal{S}^\omega$ such that $\bigcup_n A_n = X$ and $\nu(A_n) < \infty$ for all n.

Lemma 5.2 (Cantelli Theorem)**.** *Let X be a set with an outer measure μ^*. Then for every sequence $(A_n)_n$ of subsets of X with $\sum_{n=0}^{\infty} \mu^*(A_n) < \infty$, we have*

$$\mu^*\big(\limsup_n A_n\big) = 0,$$

where

$$\limsup_n A_n := \bigcap_n \bigcup_{k \geq n} A_k.$$

Proof. One has

$$\mu^*\big(\bigcap_n \bigcup_{k \geq n} A_k\big) \leq \mu^*\big(\bigcup_{k \geq m} A_k\big) \leq \sum_{k=m}^{\infty} \mu^*(A_k)$$

for every $m \in \mathbb{N}$. $\square$

Lemma 5.3. *Let X be a set with an outer measure μ^*. If $(f_n)_n$ is a sequence of $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable mappings from X into a metric space (Y, d), and $f : X \to Y$ is an arbitrary mapping with*

$$(\forall\, n \in \mathbb{N})\ \big(\mu^*([d(f_n, f) > 2^{-n}]) \leq 2^{-n}\big), \tag{5.1}$$

then[4] *f is $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable.*

Proof. Define

$$G := \big\{x \in X \ :\ \lim_{n \to \infty} f_n(x) = f(x)\big\}.$$

The functions $f_n|_G$ are each $(\mathrm{MEAS}_{\mu^*} \cap G, \mathcal{B}(Y))$-measurable, so their pointwise limit $f|_G$ is measurable, as well (see [Kal02, Lemma 1.10(ii)]). Every x with the property

$$(\exists\, n \in \mathbb{N})(\forall\, k \in \mathbb{N})\ (k < n\ \vee\ d(f_k(x), f(x)) \leq 2^{-k})$$

is in G. So

$$X \setminus G \subseteq [(\forall\, n \in \mathbb{N})(\exists\, k \in \mathbb{N})\ (k \geq n\ \wedge\ d(f_k, f) > 2^{-k})] = \limsup_n\, [d(f_n, f) > 2^{-n}].$$

By Cantelli's Theorem and (5.1), we have

$$\mu^*(X \setminus G) \leq \mu^*\big(\limsup_n [d(f_n, f) > 2^{-n}]\big) = 0.$$

So G is the complement of a μ^*-null set. In view of the completeness of $(X, \mathrm{MEAS}_{\mu^*}, \mu^*)$, the claim follows from Lemma 5.1. $\square$

[4] $[d(f_n, f) > 2^{-n}]$ denotes the set $\{x \in X\ :\ d(f_n(x), f(x)) > 2^{-n}\}$. In the following, similar expressions are to be interpreted accordingly. This is standard notation in probability theory.

5.1.3 Equivalence classes

Let X, Y be sets and μ^* an outer measure on X. An equivalence relation $\sim_{\mu^*}$ on the functions from X to Y is given by

$$f \sim_{\mu^*} g \quad :\Longleftrightarrow \quad \mu^*([f \neq g]) = 0.$$

It is in many contexts not useful to distinguish mappings that are equivalent with respect to this relation. Let $[f]_{\mu^*}$ denote the $\sim_{\mu^*}$-equivalence class of f. We will also speak more conveniently of μ^***-equivalence classes**.

If $(X, \mathcal{S}, \nu)$ is a measure space, we will use the shorthands $\sim_\nu :=\sim_{\nu^*}$ and $[f]_\nu := [f]_{\nu^*}$, where ν^* is the outer measure induced by ν. We speak of ν**-equivalence classes**.

5.1.4 Outer integrals

Let $(X, \mathcal{S}, \nu)$ be a measure space, and let $h : X \to [0; \infty]$ be an arbitrary function. We define the **outer integral** of h with respect to ν as

$$\int^* h \, d\nu := \inf \Big\{ \int g \, d\nu \ : \ g \text{ is } (\mathcal{S}, \mathcal{B}([0; \infty]))\text{-measurable, } h \leq g \Big\}.$$

We collect a number of properties of the outer integral:

Lemma 5.4. *(1) The outer integral is monotone, i.e.*

$$h_1 \leq h_2 \Rightarrow \int^* h_1 \, d\nu \leq \int^* h_2 \, d\nu.$$

(2) The outer integral is sublinear, i.e.

$$\int^* (h_1 + h_2) \, d\nu \leq \int^* h_1 \, d\nu + \int^* h_2 \, d\nu$$

and

$$\int^* th \, d\nu = t \int^* h \, d\nu$$

for all $t \in [0; \infty)$.

(3) For every $A \subseteq X$*, one has*

$$\nu^*(A) = \int^* \chi_A \, d\nu.$$

Proof. The proofs of (1) and (2) are elementary. We only prove (3). So let $A \subseteq X$ be arbitrary.

"$\geq$": Let $B \in \mathcal{S}$ with $A \subseteq B$ be arbitrary. Then χ_B is $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$-measurable with $\chi_A \leq \chi_B$. Furthermore, $\nu(B) = \int \chi_B \, d\nu$.

"$\leq$": Let g be $(\mathcal{S}, \mathcal{B}([0;\infty]))$-measurable with $\chi_A \leq g$. Put $B := g^{-1}([1;\infty])$. Then $B \in \mathcal{S}$ and $A \subseteq B$. Furthermore, $\chi_B \leq g$ and hence $\nu(B) \leq \int g \, d\nu$. □

5.1.5 Outer regularity

Let X be a topological space and let $\mathcal{S}$ be a σ-algebra on X that includes $\mathcal{B}(X)$. A measure μ on $\mathcal{S}$ is called **outer-regular** if[5]

$$(\forall\, A \in \mathcal{S}) \; \big(\inf\{\mu(G \setminus A) \;:\; G \supseteq A,\, G \text{ open}\} = 0\big).$$

It is well known that on metric spaces all finite Borel measures are outer-regular (see [Kal02, Lemma 1.34]).

We will call an outer measure μ^* on X **outer-regular** if

$$(\forall\, A \in \mathrm{MEAS}_{\mu^*}) \; \big(\inf\{\mu^*(G \setminus A) \;:\; G \supseteq A,\, G \text{ open}\} = 0\big).$$

The following lemma will be needed in the proof of Proposition 5.17 below:

Lemma 5.5. *Let X be a topological space, and let $\mathcal{S}$ be a σ-algebra on X that includes $\mathcal{B}(X)$. Let μ be an outer-regular measure on $\mathcal{S}$ and let $f : X \to [0;\infty]$ be a μ-integrable function. Then the measure ν on $\mathcal{S}$ defined by $\nu(A) := \int_A f \, d\mu$ is outer-regular.*

Proof. Let $A \in \mathcal{S}$ be arbitrary. By the outer regularity of μ, there exists a descending sequence $(G_n)_n$ of open sets such that $G_n \supseteq A$ and $\lim_n \mu(G_n \setminus A) = 0$. The set $C := \bigcap_n G_n \setminus A$ has measure 0 and so $\int_C f \, d\mu = 0$. The sequence $(f \cdot \chi_{G_n \setminus A})_n$ converges pointwise to $f \cdot \chi_C$; the sequence is furthermore dominated by the μ-integrable function f. So we can use Dominated Convergence (see [Kal02, Theorem 1.21]) to obtain

$$\lim_{n\to\infty} \nu(G_n \setminus A) = \lim_{n\to\infty} \int_{G_n \setminus A} f \, d\mu = \int_C f \, d\mu = 0.$$

□

[5] In many textbooks, a measure μ is called outer-regular if it fulfills the weaker condition that $\mu(A) = \inf\{\mu(G) \;:\; G \supseteq A,\, G \text{ open}\}$ for all $A \in \mathcal{S}$. It will be crucial for some of the results below that outer regularity is understood in the strong sense!

5.2 Three probabilistic concepts of computability

For the rest of Chapter 5, we denote by

- X, X_1 nonempty sets,
- δ, δ_1 naming systems of X, X_1, respectively,
- (Y, d, α) a computable metric space,
- μ^* an outer measure on X,
- $\mathcal{S}$ a σ-algebra on X,
- ν a measure on $(X, \mathcal{S})$,
- ν^* the outer measure induced by ν.

We furthermore put

$$\eta := \begin{cases} \nu_{\mathbb{N}} & \text{if } \mathrm{TYPE}(\delta) = \mathbb{N}, \\ [\nu_{\mathbb{N}}]^{\omega} & \text{if } \mathrm{TYPE}(\delta) = \mathbb{N}^{\omega}. \end{cases}$$

5.2.1 The local error

For any mapping $f : X \to Y$ and any $\phi :\subseteq \mathrm{TYPE}(\delta) \to \mathbb{N}$ with $\mathrm{dom}(\delta) \subseteq \mathrm{dom}(\phi)$, define the **local error**

$$\begin{aligned} &e(f, \delta, \phi, \cdot) : X \to [0; \infty], \\ &e(f, \delta, \phi, x) := \sup_{p \in \delta^{-1}\{x\}} d((\alpha \circ \phi)(p), f(x)). \end{aligned}$$

The following observation will be useful below:

Lemma 5.6. *Let $f : X \to Y$ and $\phi :\subseteq \mathrm{TYPE}(\delta) \to \mathbb{N}$ be arbitrary with $\mathrm{dom}(\delta) \subseteq \mathrm{dom}(\phi)$. Let $g :\subseteq W \to \mathrm{TYPE}(\delta)$ ($W \in \{\mathbb{N}, \mathbb{N}^{\omega}\}$) be a mapping such that $\delta \circ g$ is a naming system of X. Then*

$$e(f, \delta \circ g, \phi \circ g, x) \leq e(f, \delta, \phi, x)$$

for every $x \in X$.

Proof. For every $x \in X$, we have

$$\begin{aligned} e(f, \delta \circ g, \phi \circ g, x) &= \sup_{q \in (\delta \circ g)^{-1}\{x\}} d((\alpha \circ \phi \circ g)(q), f(x)) \\ &= \sup_{p \in \mathrm{range}(g) \cap \delta^{-1}\{x\}} d((\alpha \circ \phi)(p), f(x)) \\ &\leq e(f, \delta, \phi, x). \end{aligned}$$

□

5.2.2 The general idea

For any mapping $\phi :\subseteq \mathbb{N} \times A \to B$ (for sets A, B) and any $n \in \mathbb{N}$, we shall denote by $\phi_n :\subseteq A \to B$ the mapping given by

$$\mathrm{dom}(\phi_n) := \{a \in A \ : \ (n, a) \in \mathrm{dom}(\phi)\} \quad \text{and} \quad (\forall\, a \in \mathrm{dom}(\phi_n))\ (\phi_n(a) := \phi(n, a)).$$

The probabilistic concepts of computability that we will define below are inspired by the following simple observation:

Lemma 5.7. *A mapping $f : X \to Y$ is $(\delta, \alpha_{\mathrm{Cauchy}})$-computable if, and only if, there is a $([\nu_{\mathbb{N}}, \eta], \nu_{\mathbb{N}})$-computable $\phi :\subseteq \mathbb{N} \times \mathrm{TYPE}(\delta) \to \mathbb{N}$ such that $\mathbb{N} \times \mathrm{dom}(\delta) \subseteq \mathrm{dom}(\phi)$ and*

$$(\forall\, n \in \mathbb{N})(\forall\, x \in X)\ (e(f, \delta, \phi_n, x) \leq 2^{-n}). \tag{5.2}$$

□

In a real-word situation in which the inputs x for which we would like to compute $f(x)$ are distributed according to some measure on X, it is meaningful to replace the quantification over x in (5.2) by a probabilistic requirement on the local error. We might in particular merely require the condition $e(f, \delta, \phi_n, x) \leq 2^{-n}$ to hold

(I) for all x outside a set of measure zero, or

(II) for all x outside a set of measure not exceeding 2^{-n}, or

(III) on the average over x.

In the next three sections, we make these ideas precise.

If we look at the role of the measure in ideas (I) and (II), we notice that its role is merely to quantify the "smallness" of certain sets; the measures additivity properties are not important. So it is natural to use outer measures instead of measures for the formalization of these ideas.

Regarding idea (III), it is natural to formalize the averaging by an outer integral instead of an integral, because we are only interested in quantifying the "smallness" of the error function; for this purpose we can do without the integral's linearity. Working with outer measures and outer integrals also spares us the problem that it is a-priori not clear whether the local error is measurable with respect to the σ-algebra on which the measure is defined.[6]

5.2.3 Concept (I): Computability almost everywhere

Parker (see [Par03, Par05, Par06]) introduced the concept of *decidability up to measure zero* of a subset of Euclidean space. The following is a rather straight-forward generalization. In contrast to concepts (II) and (III), concept (I) also makes sense when the codomain is not metric.

A mapping $f : X \to X_1$ is $(\delta, \delta_1)^{\mu^*}_{\mathrm{AE}}$**-continuous (-computable)** if there is a set $N \subseteq X$ with $\mu^*(N) = 0$ such that $f|_{X\setminus N}$ is $(\delta|^{X\setminus N}, \delta_1)$-continuous (-computable).

In order to prove uniform computability results below, we need an effective representation of the $(\delta, \delta_1)^{\mu^*}_{\mathrm{AE}}$-continuous mappings. We make an observation: If N is a μ^*-null set, ϕ is a $(\delta|^{X\setminus N}, \delta_1)$-realization of $f|_{X\setminus N}$, and $g : X \to X_1$ is a mapping such that $f \sim_{\mu^*} g$, then ϕ is also a continuous $(\delta|^{X\setminus N'}, \delta_1)$-realization of $g|_{X\setminus N'}$ for $N' := N \cup [f \neq g]$, and we have $\mu^*(N') = 0$. So if f is $(\delta, \delta_1)^{\mu^*}_{\mathrm{AE}}$-continuous, then all elements of $[f]_{\mu^*}$ are also $(\delta, \delta_1)^{\mu^*}_{\mathrm{AE}}$-continuous via the same realization. This suggests to define a representation of μ^*-equivalence classes of mappings. A representation of the class $\Lambda(\delta \to \delta_1)^{\mu^*}_{\mathrm{AE}}$ of all μ^*-equivalence classes of $(\delta, \delta_1)^{\mu^*}_{\mathrm{AE}}$-continuous mappings is given by

$$[\delta \to \delta_1]^{\mu^*}_{\mathrm{AE}}(p) = [f]_{\mu^*} \quad :\Longleftrightarrow \quad \left[\begin{array}{l}\text{there is a set } N \subseteq X \text{ with } \mu^*(N) = 0 \text{ such that} \\ \left[\delta|^{X\setminus N} \to \delta_1\right]_{\mathrm{TOT}}(p) = f|_{X\setminus N}.\end{array}\right.$$

We have already seen that this definition is independent of the choice of the representative f. The following lemma ensures that the representation is in fact single-valued:

Lemma 5.8. *If $[f]_{\mu^*} \neq [g]_{\mu^*}$, then there is an $\varepsilon > 0$ such that $\mu^*([])$*

Let us introduce the following shorthands: $f : X \to X_1$ is $(\delta, \delta_1)^{\nu}_{\mathrm{AE}}$**-continuous (-computable)** if f is $(\delta, \delta_1)^{\nu^*}_{\mathrm{AE}}$-continuous (-computable). A representation of the class $\Lambda(\delta \to \delta_1)^{\nu}_{\mathrm{AE}}$ of all ν-equivalence classes of $(\delta, \delta_1)^{\nu}_{\mathrm{AE}}$-continuous mappings is given by

$$[\delta \to \delta_1]^{\nu}_{\mathrm{AE}} := [\delta \to \delta_1]^{\nu^*}_{\mathrm{AE}}.$$

[6] Typically, the local error *is* measurable; see [Bos08c]. The question of measurability of the local error, however, will not play any role in the theory that is developed below.

5.2.4 Concept (II): Computable approximation

The definitions in this subsection generalize a definition given by Ko (cf. [Ko91, Definition 5.10]), who studied the special case of functions from $\mathbb{R}$ to $\mathbb{R}$.

Let $f : X \to Y$ be a mapping. A mapping $\phi :\subseteq \mathbb{N} \times \mathrm{TYPE}(\delta) \to \mathbb{N}$ is a $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$**-realization** of f if $\mathbb{N} \times \mathrm{dom}(\delta) \subseteq \mathrm{dom}(\phi)$ and

$$(\forall\, n \in \mathbb{N}) \left(\mu^*([e(f, \delta, \phi_n, \cdot) > 2^{-n}]) \leq 2^{-n}\right).$$

f is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$**-continuous (-computable)** if it has a continuous (computable) $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-realization.

We observe that if ϕ is a $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-realization of f, then ϕ is also a $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-realization of any $g \in [f]_{\mu^*}$, because

$$\begin{aligned} &\mu^*([e(f, \delta, \phi_n, \cdot) > 2^{-n}]) = \mu^*([e(f, \delta, \phi_n, \cdot) > 2^{-n}] \setminus [f \neq g]) \\ &= \mu^*([e(g, \delta, \phi_n, \cdot) > 2^{-n}] \setminus [f \neq g]) = \mu^*([e(g, \delta, \phi_n, \cdot) > 2^{-n}]). \end{aligned}$$

A representation of the class $\Lambda(\delta \to \alpha)^{\mu^*}_{\mathrm{APP}}$ of all μ^*-equivalence classes of $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-continuous mappings is given by

$$[\delta \to \alpha]^{\mu^*}_{\mathrm{APP}}(p) = [f]_{\mu^*} \quad :\Longleftrightarrow \quad [[\nu_{\mathbb{N}}, \eta] \to \nu_{\mathbb{N}}](p) \text{ contains a } (\delta, \alpha)^{\mu^*}_{\mathrm{APP}}\text{-realization of } f.$$

$f : X \to Y$ is $(\delta, \alpha)^{\nu}_{\mathrm{APP}}$**-continuous (-computable)** if f is $(\delta, \alpha)^{\nu^*}_{\mathrm{APP}}$-continuous (-computable). A representation of the class $\Lambda(\delta \to \alpha)^{\nu}_{\mathrm{APP}}$ of all ν-equivalence classes of $(\delta, \alpha)^{\nu}_{\mathrm{APP}}$-continuous mappings is given by

$$[\delta \to \alpha]^{\nu}_{\mathrm{APP}} := [\delta \to \alpha]^{\nu^*}_{\mathrm{APP}}.$$

The definition just given requires $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-realizations to be defined on all of $\mathbb{N} \times \mathrm{dom}(\delta)$, i.e. a Turing machine that implements such a realization must halt on every (properly encoded) input from $\mathbb{N} \times \mathrm{dom}(\delta)$ and put out an element of $\mathbb{N}$. Concerning this definition, we assent to the following statement of Parker (see [Par03, p. 8]):

> Why require a machine that always halts? Assuming we have a machine that sometimes gives incorrect output, the epistemological situation would seem no worse if in principle that machine could also fail to halt, but with probability zero.

This suggests a combination of concepts (I) and (II): A mapping $\phi :\subseteq \mathbb{N} \times \mathrm{TYPE}(\delta) \to \mathbb{N}$ is a $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$**-realization** of f if there exists a set $N \subseteq X$ with $\mu^*(N) = 0$ such that

$$\mathbb{N} \times \mathrm{dom}(\delta|^{X \setminus N}) \subseteq \mathrm{dom}(\phi)$$

and

$$(\forall\, n \in \mathbb{N})\ \big(\mu^*(\{x \in X \setminus N \ :\ e(f, \delta, \phi_n, x) > 2^{-n}\}) \le 2^{-n}\big).$$

f is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$**-continuous (-computable)** if it has a continuous (computable) $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$-realization.

If ϕ is a $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$-realization of f, then ϕ is also a $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$-realization of any $g \in [f]_{\mu^*}$. A representation of the class $\Lambda(\delta \to \alpha)^{\mu^*}_{\mathrm{APP/AE}}$ of all μ^*-equivalence classes of $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$-continuous mappings is given by

$$[\delta \to \alpha]^{\mu^*}_{\mathrm{APP/AE}}(p) = [f]_{\mu^*} \quad :\Longleftrightarrow \quad [[\nu_{\mathbb{N}}, \eta] \to \nu_{\mathbb{N}}](p) \text{ contains a } (\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}\text{-realization of } f.$$

In analogy to above, we also define a mapping f to be $(\delta, \alpha)^{\nu}_{\mathrm{APP/AE}}$**-continuous (-computable)** if it is $(\delta, \alpha)^{\nu^*}_{\mathrm{APP/AE}}$-continuous (-computable). $\Lambda(\delta \to \alpha)^{\nu}_{\mathrm{APP/AE}}$ denotes the class of all ν-equivalence classes of $(\delta, \alpha)^{\nu}_{\mathrm{APP/AE}}$-continuous mappings. This class is represented by $[\delta \to \alpha]^{\nu}_{\mathrm{APP/AE}} := [\delta \to \alpha]^{\nu^*}_{\mathrm{APP/AE}}$.

5.2.5 Concept (III): Computability in the mean

We now come to a notion that is the result of working out a proposal made in the talk [Her05] by Hertling and has apparently not been treated in the literature before.

Let $f : X \to Y$ be a mapping. A mapping $\Phi :\subseteq \mathbb{N} \times \mathrm{TYPE}(\delta) \to \mathbb{N}$ is a $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$**-realization** of f if

$$\mathbb{N} \times \mathrm{dom}(\delta) \subseteq \mathrm{dom}(\Phi)$$

and

$$(\forall\, n \in \mathbb{N})\ \Big(\int^* e(f, \delta, \Phi_n, x)\, \nu(dx) \le 2^{-n}\Big).$$

f is $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$**-continuous (-computable)** if it has a continuous (computable) $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-realization.

It is again easy to see that a $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-realization of a mapping f is also a $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-realization of all elements of $[f]_{\nu}$. A representation of the class $\Lambda(\delta \to \alpha)^{\nu}_{\mathrm{MEAN}}$ of all ν-equivalence classes of $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-continuous mappings is given by

$$[\delta \to \alpha]^{\nu}_{\mathrm{MEAN}}(p) = [f]_{\nu} \quad :\Longleftrightarrow \quad [[\nu_{\mathbb{N}}, \eta] \to \nu_{\mathbb{N}}](p) \text{ contains a } (\delta, \alpha)^{\nu}_{\mathrm{MEAN}}\text{-realization of } f.$$

A mapping $\Phi :\subseteq \mathbb{N} \times \mathrm{TYPE}(\delta) \to \mathbb{N}$ is a $(\delta, \alpha)^{\nu}_{\mathrm{MEAN/AE}}$**-realization** of f if there exists a ν-null set $N \subseteq X$ such that

$$\mathbb{N} \times \mathrm{dom}(\delta|^{X \setminus N}) \subseteq \mathrm{dom}(\Phi)$$

and

$$(\forall\, n \in \mathbb{N})\ \Big(\int^*_{X \setminus N} e(f, \delta, \Phi_n, x)\, \nu(dx) \le 2^{-n}\Big).$$

f is $(\delta, \alpha)^\nu_{\text{MEAN/AE}}$**-continuous (-computable)** if it has a continuous (computable) $(\delta, \alpha)^\nu_{\text{MEAN/AE}}$-realization. A representation of the class $\Lambda(\delta \to \alpha)^\nu_{\text{MEAN/AE}}$ of all ν-equivalence classes of $(\delta, \alpha)^\nu_{\text{MEAN/AE}}$-continuous mappings is given by

$$[\delta \to \alpha]^\nu_{\text{MEAN/AE}}(p) = [f]_\nu \quad :\Longleftrightarrow \quad [[\nu_{\mathbb{N}}, \eta] \to \nu_{\mathbb{N}}](p) \text{ contains a } (\delta, \alpha)^\nu_{\text{MEAN/AE}}\text{-realization of } f.$$

The notion of MEAN-computability just defined has a property that one would expect any reasonable notion of "computability in the mean" to have: Imagine a real-world situation in which a sequence of measurements it made. We assume that it is known that the outcomes of these measurements are independent identically distributed according to a certain probability law. We have the task to compute the same function f on each of the measured values. If f is "computable in the mean", then there should be an approximation algorithm for f whose error is small if one considers the arithmetic mean over "a large number" of measurements. In fact, we have the following result:

Proposition 5.9 (Strong Law of Large Numbers for MEAN-computability)**.** *Suppose that ν is a probability measure. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(w_i)_i$ be a sequence of mappings $w_i : \Omega \to \mathrm{dom}(\delta)$ such that the mappings $\delta \circ w_i$ are independent ν-distributed random variables. Let $f : X \to Y$ be a mapping which has a $(\delta, \alpha)^\nu_{\text{MEAN}}$-realization Φ. Then for every $n \in \mathbb{N}$, one has*

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} e_i(\omega) \leq 2^{-n}$$

for P-almost every ω, where

$$e_i(\omega) := d((\alpha \circ \Phi_n \circ w_i)(\omega), (f \circ \delta \circ w_i)(\omega)).$$

Proof. It follows directly from the definition of the local error that

$$(\forall\, i \in \mathbb{N})(\forall\, \omega \in \Omega)\ (e_i(\omega) \leq e(f, \delta, \Phi_n, (\delta \circ w_i)(\omega))).$$

As Φ is a MEAN-realization, and by the definition of the outer integral, there is a sequence $(g_k)_k$ of measurable functions $g_k : X \to [0; \infty]$ such that

$$(\forall\, k \in \mathbb{N})(\forall\, x \in X)\ (e(f, \delta, \Phi_n, x) \leq g_k(x))$$

and

$$(\forall\, k \in \mathbb{N})\ \Big(\int g_k \, d\nu \leq 2^{-n} + 2^{-k}\Big).$$

The Strong Law of Large Numbers (see [Kal02, Theorem 4.23]) yields that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} g_k((\delta \circ w_i)(\omega)) = \int g_k \, d\nu$$

for every k and P-almost every ω. So the following estimate holds for every k and P-almost every ω:

$$\limsup_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} e_i(\omega) \le \limsup_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} g_k((\delta\circ w_i)(\omega)) = \int g_k\, d\nu \le 2^{-n} + 2^{-k}$$

Intersecting over k yields the claim. $\square$

We close this section with the following lemma which is a simple consequence of Lemma 5.6. Its analogue for plain computability can be found in [Wei00, Exercise 3.3.13].

Lemma 5.10. *Suppose that δ' is another naming system of X, and $(\sim, \square)$ is one of (APP, μ^*), $(\text{APP/AE}, \mu^*)$, (MEAN, ν), $(\text{MEAN/AE}, \nu)$.*

(1) If $\delta' \le_t \delta$, then every $(\delta,\alpha)^{\square}_{\sim}$-continuous mapping is $(\delta',\alpha)^{\square}_{\sim}$-continuous and $[\delta\to\alpha]^{\square}_{\sim} \le_t [\delta'\to\alpha]^{\square}_{\sim}$.

(2) If $\delta' \le \delta$, then $[\delta\to\alpha]^{\square}_{\sim} \le [\delta'\to\alpha]^{\square}_{\sim}$.

(3) If $\delta' \equiv_t \delta$, then $[\delta\to\alpha]^{\square}_{\sim} \equiv_t [\delta'\to\alpha]^{\square}_{\sim}$.

(4) If $\delta' \equiv \delta$, then $[\delta\to\alpha]^{\square}_{\sim} \equiv [\delta'\to\alpha]^{\square}_{\sim}$.

Proof. Items (3) and (4) follow immediately from items (1) and (2), respectively. We show (1) and (2). Let g be a continuous (δ',δ)-realization of id_X, i.e. $\delta' = \delta\circ g$. Let

$$\phi :\subseteq \mathbb{N}\times \mathrm{TYPE}(\delta) \to \mathbb{N}$$

and $f : X \to Y$ be arbitrary. If ϕ is a $(\delta|^A, \alpha)^{\square}_{\sim}$-realization of $f|_A$ for a set $A \subseteq X$, then ϕ^g with

$$\phi^g(n,p) := \phi(n, g(p)), \qquad n\in\mathbb{N},\, p\in \mathrm{TYPE}(\delta),$$

is a $(\delta'|^A, \alpha)^{\square}_{\sim}$-realization of $f|_A$, because Lemma 5.6 yields that

$$e(f,\delta',\phi^g_n, x) = e(f, \delta\circ g, \phi_n\circ g, x) \le e(f,\delta,\phi_n,x)$$

for every $n\in\mathbb{N}$ and $x\in A$. If ϕ is continuous, so is ϕ^g. This yields the first claim in (1). (Choose $A = X$ for $\sim \in \{\text{APP}, \text{MEAN}\}$ and $A = X\setminus N$ with a suitable μ^*-null set N for $\sim\in\{\text{APP/AE}, \text{MEAN/AE}\}$.) If g is continuous, so is the map $\phi\mapsto\phi^g$; this yields the second claim in (1). If g is computable, so is the map $\phi\mapsto\phi^g$; this yields (2). $\square$

5.3 Representation theorems

The Representation Theorem (see Theorem 2.6) gives a characterization of the mappings that are continuous with respect to certain naming systems. In the spirit of the Representation Theorem, we now seek for connections between classical properties of a mapping and its probabilistic relative continuity as defined in the preceding section.

5.3.1 Representation Theorem for AE-continuity

Proposition 5.11. *Assume that X and X_1 are endowed with topologies with respect to which δ and δ_1 are admissible. Then a mapping $f : X \to Y$ is $(\delta, \delta_1)^{\mu^*}_{\mathrm{AE}}$-continuous if, and only if, there is a set $N \subseteq X$ with $\mu^*(N) = 0$ such that $f|_{X\setminus N}$ is sequentially continuous.*

Proof. By [Sch02c, Subsection 4.1], $\delta|^{X\setminus N}$ is an admissible representation of $X \setminus N$ for any subset N of X. The claim hence follows directly from Theorem 2.6. □

5.3.2 Representation Theorem for APP(/AE)-continuity

Denote by $\sigma(\delta^{-1})$ the σ-algebra on X that is generated by $\{\delta(U) \ : \ U \text{ open in } \mathrm{TYPE}(\delta)\}$.

Proposition 5.12. *Assume that $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\mu^*}$. Then every $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$-continuous $f : X \to Y$ is $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable.*

Proof. It follows from Lemma 5.1 that it is sufficient to prove the claim for $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-continuous f. Let ϕ be a continuous $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-realization of f. For all $n, m \in \mathbb{N}$, put $A_{n,m} := \phi_n^{-1}\{m\}$. Then every $A_{n,m}$ is open in $\mathrm{dom}(\delta)$, and $\mathrm{dom}(\delta) \subseteq \bigcup_m A_{n,m}$ for every n. The assumption $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\mu^*}$ implies that all sets $D_{n,m} := \delta(A_{n,m})$ are in MEAS_{μ^*}. Define

$$c(n, x) := \min\{m \in \mathbb{N} \ : \ x \in D_{n,m}\},$$
$$f_n(x) := \alpha(c(n, x)).$$

For every $a \in \mathcal{R}_\alpha$, every $x \in X$, and every $n \in \mathbb{N}$, we have

$$f_n(x) = a \ \Leftrightarrow \ (\exists\, m \in \mathbb{N})\ (\alpha(m) = a,\ c(n, x) = m) \ \Leftrightarrow \ x \in \bigcup_{m\in\alpha^{-1}\{a\}} \big(D_{n,m} \setminus \bigcup_{k<m} D_{n,k}\big).$$

So $f_n^{-1}\{a\} \in \mathrm{MEAS}_{\mu^*}$ for every $a \in \mathcal{R}_\alpha$, which yields that the f_n are $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable. For every $x \in X$, we have that $f_n(x)$ is the output of ϕ_n on a certain δ-name of x; it hence follows from the definition of the local error that

$$d(f_n(x), f(x)) \leq e(f, \delta, \phi_n, x)$$

for all $x \in X$, so

$$\mu^*([d(f, f_n) > 2^{-n}]) \leq \mu^*([e(f, \delta, \phi_n, \cdot) > 2^{-n}]) \leq 2^{-n}$$

for every $n \in \mathbb{N}$. The claim now follows with Lemma 5.3. □

Proposition 5.13. *Suppose that X is endowed with a topology with respect to which δ is continuous and μ^* is outer-regular. Then every $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable $f : X \to Y$ is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-continuous.*

Proof. For all $m, n \in \mathbb{N}$, put

$$A_{m,n} := f^{-1}(B(\alpha(m), 2^{-n})).$$

Note that $X = \bigcup_m A_{m,n}$. By the outer regularity of μ^*, there are open sets $G_{m,n}$ with $A_{m,n} \subseteq G_{m,n}$ and $\mu^*(G_{m,n} \setminus A_{m,n}) \leq 2^{-(n+m+1)}$. Now for every $n \in \mathbb{N}$, there is a continuous "selector" $c_n : \mathrm{dom}(\delta) \to \mathbb{N}$ such that $\delta(p) \in G_{c_n(p),n}$ for every $p \in \mathrm{dom}(\delta)$. Put $\phi(n, p) := c_n(p)$. For every $x \in X$ and every $n \in \mathbb{N}$, we have

$$\begin{aligned}
& x \in X \setminus \bigcup_{m \in \mathbb{N}} (G_{m,n} \setminus A_{m,n}) \\
\Rightarrow\ & (\forall\, m \in \mathbb{N})\ (x \in G_{m,n} \Rightarrow x \in A_{m,n}) \\
\Rightarrow\ & (\forall\, p \in \delta^{-1}\{x\})(\forall\, m \in \mathbb{N})\ (c_n(p) = m \Rightarrow x \in A_{m,n}) \\
\Rightarrow\ & (\forall\, p \in \delta^{-1}\{x\})\ (x \in A_{c_n(p),n}) \\
\Rightarrow\ & (\forall\, p \in \delta^{-1}\{x\})\ (d((\alpha \circ c_n)(p), f(x)) < 2^{-n}) \\
\Rightarrow\ & e(f, \delta, \phi_n, x) \leq 2^{-n}.
\end{aligned}$$

This implies

$$\mu^*([e(f, \delta, \phi_n, \cdot) > 2^{-n}]) \leq \mu^*\big(\bigcup_{m \in \mathbb{N}} (G_{m,n} \setminus A_{m,n})\big) \leq 2^{-n}$$

for every n. □

Combining the last two propositions yields the following corollary, which should apply in most situations of practical interest:

Corollary 5.14. *Suppose that X is endowed with a topology with respect to which δ is continuous and μ^* is outer-regular. Also suppose $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\mu^*}$. Then for every mapping $f : X \to Y$, the following statements are equivalent:*

(1) f is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-continuous.

(2) f is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$-continuous.

(3) f is $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable.

□

The condition $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\mu^*}$ is fulfilled, for example, if $\delta = \vartheta_{\mathrm{std}}$ for a computable T_0-space (X, ϑ) such that $\tau_\vartheta \subseteq \mathrm{MEAS}_{\mu^*}$, because ϑ_{std} is an open mapping (see Lemma 2.9).

5.3.3 Representation Theorem for MEAN(/AE)-continuity

The next result follows as a simple combination of Proposition 5.12 and Proposition 5.19; although the latter will be proved only below, we think that the corollary should be stated already here:

Corollary 5.15. *Assume that $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\nu^*}$. If $f : X \to Y$ is $(\delta, \alpha)^{\nu}_{\mathrm{MEAN/AE}}$-continuous, then f is $(\mathrm{MEAS}_{\nu^*}, \mathcal{B}(Y))$-measurable.* □

We will see below (Proposition 5.25.2) that conditions such as those of Proposition 5.13 (δ continuous, ν outer-regular, f measurable) are not sufficient to ensure MEAN-continuity. The next natural step is to consider integrable f. This makes sense only if Y is a normed space. We thus make a number of assumptions that shall be valid in this subsection:

- Y is a normed space with norm $\|\cdot\|$, and d is the metric induced by the norm.
- $0 \in \mathcal{R}_\alpha$.[7]
- X is endowed with a topology.

Proposition 5.16. *Suppose that δ is open and ν^* is locally finite. If a mapping $f : X \to Y$ is $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-continuous, then $\|f\|$ is locally outer-integrable with respect to ν, i.e. for every $x \in X$ there is an open neighbourhood $G \subseteq X$ of x such that $\int^*_G \|f\|\, d\nu < \infty$.*

Proof. Let Φ be a continuous $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-realization of f. Let $x_0 \in X$ be arbitrary, and let p be an arbitrary δ-name of x_0. Φ_0 is constantly equal to $\Phi_0(p)$ on an open (in $\mathrm{dom}(\delta)$) neighbourhood $U \subseteq \mathrm{dom}(\delta)$ of p. Put $a := (\alpha \circ \Phi_0)(p)$. By the definition of the local error, we have

$$(\forall\, x \in \delta(U))\ (e(f, \delta, \Phi_0, x) \geq \|a - f(x)\|).$$

$\delta(U)$ is open, and by the local finiteness of ν^*, we can find an open neighbourhood $G \subseteq \delta(U)$ of x_0 such that $\nu^*(G) < \infty$. We finally have

$$\begin{aligned} 1 &\geq \int^* e(f, \delta, \Phi_0, x)\, \nu(dx) \geq \int^*_G e(f, \delta, \Phi_0, x)\, \nu(dx) \geq \int^*_G \|a - f(x)\|\, \nu(dx) \\ &\geq \int^*_G \|f\|\, d\nu - \int^*_G \|a\|\, d\nu = \int^*_G \|f\|\, d\nu - \nu^*(G)\|a\|. \end{aligned}$$

and hence

$$\int^*_G \|f\|\, d\nu \leq 1 + \nu^*(G)\|a\| < \infty.$$

□

[7] Note that this is fulfilled if there is a computable normed space $(Y, \|\cdot\|, e)$ such that $\alpha = \alpha^e$.

Proposition 5.17. *Suppose that δ is continuous, $\mathcal{B}(X) \subseteq \mathcal{S}$, ν is outer-regular, f is $(\mathcal{S}, \mathcal{B}(Y))$-measurable, and $\|f\|$ is locally ν-integrable. Then f is $(\delta, \alpha)^{\nu}_{\text{MEAN}}$-continuous.*

Proof. We first assume that $\|f\|$ is integrable over the whole space. For all $m, n \in \mathbb{N}$ put

$$A_{m,n} := \begin{cases} f^{-1}\big(\, B(\alpha(m), \min(2^{-n}, \|\alpha(m)\|/2))\,\big) & \text{if } \alpha(m) \neq 0 \\ f^{-1}\{0\} & \text{otherwise.} \end{cases}$$

For every $x \in X \setminus \{0\}$ there is an $a \in \mathcal{R}_\alpha$ with $d(x, a) < \min(2^{-n}, \|x\|/3)$; then

$$\|x\|/3 = (\|x\| - \|x\|/3)/2 \leq (\|x\| - \|a - x\|)/2 \leq \|a\|/2,$$

and hence $x \in B(a, \min(2^{-n}, \|a\|/2))$. This yields $X = \bigcup_m A_{m,n}$. Put

$$C_{m,n} := A_{m,n} \setminus \bigcup_{k<m} A_{k,n}$$

and

$$g_n := \sum_{m=0}^{\infty} \alpha(m) \chi_{C_{m,n}}.$$

Then $\|f - g_n\| \leq \min(2^{-n}, \|g_n\|/2)$. This yields both that $(g_n)_n$ converges pointwise to f and that

$$\|f - g_n\| = 2\|f - g_n\| - \|f - g_n\| \leq \|g_n\| - \|f - g_n\| \leq \|f\|.$$

The assumption that $\|f\|$ is integrable and Dominated Convergence now yield that

$$\lim_{n\to\infty} \int \|f - g_n\| \, d\nu = 0.$$

By transition to a subsequence, we can assume that

$$\int \|f - g_n\| \, d\nu < 2^{-(n+1)} \tag{5.3}$$

for all $n \in \mathbb{N}$. The measures ν_n on $\mathcal{S}$ defined by

$$\nu_n(A) := \int_A \|g_n\| \, d\nu$$

are outer-regular by Lemma 5.5. So there are open sets $G_{m,n}$ with $G_{m,n} \supseteq C_{m,n}$,

$$\nu(G_{m,n} \setminus C_{m,n}) \leq 2^{-(n+m+3)} \cdot (\max\{1, \|\alpha(m)\|\})^{-1}, \tag{5.4}$$

and

$$\nu_n(G_{m,n} \setminus C_{m,n}) \leq 2^{-(n+m+3)}. \tag{5.5}$$

For every $n \in \mathbb{N}$, there is a continuous $m_n : \mathrm{dom}(\delta) \to \mathbb{N}$ such that $\delta(p) \in G_{m_n(p),n}$ for every $p \in \mathrm{dom}(\delta)$. Put $\Phi(n,p) := m_n(p)$ and $E_n := \bigcup_m (G_{m,n} \setminus C_{m,n})$. Note that $e(g_n, \delta, \Phi_n, x) = 0$ if $x \notin E_n$; if $x \in E_n$, then

$$e(g_n, \delta, \Phi_n, x) \le \sup \big\{ \|\alpha(m) - g_n(x)\| \ : \ m \in \mathbb{N},\, x \in G_{m,n} \setminus C_{m,n} \big\};$$

together this justifies the estimate

$$e(g_n, \delta, \Phi_n, x) \le \sum_{m=0}^{\infty} \chi_{G_{m,n} \setminus C_{m,n}}(x) \|\alpha(m) - g_n(x)\|. \tag{5.6}$$

By means of estimates (5.3), (5.4), (5.5), (5.6), we get

$$\begin{aligned}
\int^* e(f, \delta, \Phi_n, x)\, \nu(dx) &\le \int \|f - g_n\|\, d\nu + \int^* e(g_n, \delta, \Phi_n, x)\, \nu(dx) \\
&\le 2^{-(n+1)} + \sum_{m=0}^{\infty} \int_{G_{m,n} \setminus C_{m,n}} \|\alpha(m) - g_n(x)\| \nu(dx) \\
&\le 2^{-(n+1)} + \sum_{m=0}^{\infty} \nu(G_{m,n} \setminus C_{m,n}) \|\alpha(m)\| + \sum_{m=0}^{\infty} \nu_n(G_{m,n} \setminus C_{m,n}) \\
&\le 2^{-n}.
\end{aligned}$$

So Φ is a continuous $(\delta, \alpha)^{\nu}_{\text{MEAN}}$-realization of f.

Now assume that $\|f\|$ is only locally integrable. Remember that X is Lindelöf (see Lemma (2.5)). There hence is a countable open cover $(G_\ell)_\ell$ of X, such that $\|f\|$ is integrable on each G_ℓ. By the first part of the proof, each mapping $f|_{G_\ell}$ is $(\delta|^{G_\ell}, \alpha)^{\nu}_{\text{MEAN}}$-continuous; let $\Phi^{(\ell)}$ be the corresponding realization. Let $c : \mathrm{dom}(\delta) \to \mathbb{N}$ be a continuous selector such that $\delta(p) \in G_{c(p)}$ for every $p \in \mathrm{dom}(\delta)$. Now put

$$\Phi(n,p) := \Phi^{(c(p))}(n + c(p) + 1, p).$$

One then has the estimate:

$$\begin{aligned}
\int^* e(f, \delta, \Phi_n, x)\, \nu(dx) &\le \int^* \sup_\ell \chi_{G_\ell}(x) e(f, \delta, \Phi^{(\ell)}_{n+\ell+1}, x)\, \nu(dx) \\
&\le \sum_{\ell=0}^{\infty} \int^*_{G_\ell} e(f, \delta, \Phi^{(\ell)}_{n+\ell+1}, x)\, \nu(dx) \\
&\le 2^{-n}.
\end{aligned}$$

□

The following corollary should apply in most situations of practical interest:

Corollary 5.18. *Suppose that the topology of X is countably-based T_0, δ is admissible, $\mathcal{B}(X) \subseteq \mathcal{S}$, ν is locally finite and outer-regular, and $f : X \to Y$ is $(\mathcal{S}, \mathcal{B}(Y))$-measurable. Then f is $(\delta, \alpha)^\nu_{\mathrm{MEAN}}$-continuous if, and only if, $\|f\|$ is locally integrable with respect to ν.*

Proof. The "if" direction follows directly by Proposition 5.17. For the "only if" direction, first recall that an admissible representation of a countably-based T_0-space is continuously equivalent to an open standard representation of that space (see [Sch02c, Section 2.2]). By Lemma 5.10, it is hence sufficient to prove the claim for open δ. It then follows directly from Proposition 5.16. □

5.4 Mutual relations between the probabilistic computability notions

5.4.1 Reductions

We will now clarify the mutual relations between the concepts defined above. The first proposition sums up the cases in which there is a computable reduction of one representation to the other. For formal reasons, we define a representation $[\delta \to \delta_1]^{\mu^*}$ of the class $\Lambda(\delta \to \delta_1)^{\mu^*}$ of all μ^*-equivalence classes of mappings $f : X \to X_1$ that have a (δ, δ_1)-continuous representative:

$$[\delta \to \delta_1]^{\mu^*}(p) = [f]_{\mu^*} \quad :\Longleftrightarrow \quad [\delta \to \delta_1](p) \cap [f]_{\mu^*} \neq \emptyset.$$

Proposition 5.19. *(1)* $\Lambda(\delta \to \delta_1)^{\mu^*} \subseteq \Lambda(\delta \to \delta_1)^{\mu^*}_{\mathrm{AE}}$ *and* $[\delta \to \delta_1]^{\mu^*} \leq [\delta \to \delta_1]^{\mu^*}_{\mathrm{AE}}$.

(2) $\Lambda(\delta \to \alpha_{\mathrm{Cauchy}})^{\mu^*}_{\mathrm{AE}} \subseteq \Lambda(\delta \to \alpha)^{\mu^*}_{\mathrm{APP/AE}}$ *and* $[\delta \to \alpha_{\mathrm{Cauchy}}]^{\mu^*}_{\mathrm{AE}} \leq [\delta \to \alpha]^{\mu^*}_{\mathrm{APP/AE}}$.

(3) $\Lambda(\delta \to \alpha)^{\nu}_{\mathrm{MEAN}} \subseteq \Lambda(\delta \to \alpha)^{\nu}_{\mathrm{APP}}$ *and* $[\delta \to \alpha]^{\nu}_{\mathrm{MEAN}} \leq [\delta \to \alpha]^{\nu}_{\mathrm{APP}}$.

(4) $\Lambda(\delta \to \alpha)^{\nu}_{\mathrm{MEAN/AE}} \subseteq \Lambda(\delta \to \alpha)^{\nu}_{\mathrm{APP/AE}}$ *and* $[\delta \to \alpha]^{\nu}_{\mathrm{MEAN/AE}} \leq [\delta \to \alpha]^{\nu}_{\mathrm{APP/AE}}$.

Proof. For (1): A realization of the left-hand type is also a realization of the right-hand type.
For (2): Let a $[\delta \to \alpha_{\mathrm{Cauchy}}]^{\mu^*}_{\mathrm{AE}}$-name of some $[f]_{\mu^*}$ be given. This name provides us with the information to compute a mapping $\phi' :\subseteq \mathrm{TYPE}(\delta) \to \mathbb{N}^\omega$ that is a $(\delta|^{X \setminus N}, \alpha_{\mathrm{Cauchy}})$-realization of $f|_{X \setminus N}$ for a suitable $N \subseteq X$ with $\mu^*(N) = 0$. We can of course compute the mapping ϕ with $\phi(n, p)$ being the n-th element of the α_{Cauchy}-name $\phi'(p)$ for every $p \in \mathrm{dom}(\phi')$. By the definition of the Cauchy representation, $\phi(n, p)$ is hence an α-name of a 2^{-n}-approximation of $(f \circ \delta)(p)$ whenever $p \in \delta^{-1}(X \setminus N)$. It is now easy to see that ϕ is a $(\delta \to \alpha)^{\mu^*}_{\mathrm{APP/AE}}$-realization of $[f]_{\mu^*}$. $\phi' \mapsto \phi$ is the desired computable reduction.

For (3) and (4): Outer integrals fulfill the following version of Markov's inequality for every $h : X \to [0; \infty]$ and $\varepsilon > 0$:

$$\int^* h \, d\nu \geq \int^* \varepsilon \cdot \chi_{[h>\varepsilon]} d\nu = \varepsilon \cdot \nu^*([h > \varepsilon]).$$

Let $f : X \to Y$ be arbitrary. If $\Phi :\subseteq \mathbb{N} \times \mathrm{TYPE}(\delta) \to \mathbb{N}$ is a $[\delta|^A \to \alpha]^\nu_{\mathrm{MEAN}}$-realization of $f|_A$ for some set $A \subseteq X$ (here $A = X$ for item (3), and A is the complement of a set N with $\nu(N) = 0$ for item (4)), then ϕ with $\phi(n,p) := \Phi(2n,p)$ is a $[\delta|^A \to \alpha]^\nu_{\mathrm{MEAN}}$-realization of $f|_A$, because, by Markov's inequality,

$$\nu^*(\{x \in A \ : \ e(f, \delta, \Phi_{2n}, x) > 2^{-n}\}) \leq 2^n \int^*_A e(f, \delta, \Phi_{2n}, x) \, d\nu \leq 2^{-n}.$$

We have that the computable map $\Phi \mapsto \phi$ is a reduction from $[\delta \to \alpha]^\nu_{\mathrm{MEAN}}$ to $[\delta \to \alpha]^\nu_{\mathrm{APP}}$ and from $[\delta \to \alpha]^\nu_{\mathrm{MEAN/AE}}$ to $[\delta \to \alpha]^\nu_{\mathrm{APP/AE}}$. $\square$

We next give conditions under which MEAN- and APP-computability coincide:

Proposition 5.20. *Suppose that Y is a normed space, the mapping $a \mapsto \|a\|$ for $a \in \mathcal{R}_\alpha$ is $(\alpha, \rho_\mathbb{R})$-computable, and $\nu(X) < \infty$. If $[f]_\nu \in \Lambda(\delta \to \alpha)^\nu_{\mathrm{APP}}$ and $N \in \mathbb{N}$ are such that $\|f\| \leq N$ ν-a.e., then $[f]_\nu \in \Lambda(\delta \to \alpha)^\nu_{\mathrm{MEAN}}$. The mapping $([f]_\nu, N) \mapsto [f]_\nu$ for $[f]_\nu \in \Lambda(\delta \to \alpha)^\nu_{\mathrm{APP}}$ and $N \in \mathbb{N}$ with $\|f\| < N$ ν-a.e. is $([[\delta \to \alpha]^\nu_{\mathrm{APP}}, \nu_\mathbb{N}], [\delta \to \alpha]^\nu_{\mathrm{MEAN}})$-computable.*

Proof. We need to demonstrate how to compute a $(\delta, \alpha)^\nu_{\mathrm{MEAN}}$-realization Φ of a mapping f from any given $(\delta, \alpha)^\nu_{\mathrm{APP}}$-realization ϕ of f and any given N with $\|f\| < N$ ν-a.e. We can assume $N > 0$. Fix an $m_0 \in \mathbb{N}$ such that $\|\alpha(m_0)\| \leq N$. For any given $(n,p) \in \mathrm{dom}(\phi)$, we can semidecide the conditions $\|(\alpha \circ \phi)(n,p)\| < 3N + 1$ and $\|(\alpha \circ \phi)(n,p)\| > 3N$; we can hence compute a function $\phi' : \mathbb{N} \times \mathrm{TYPE}(\delta) \to \mathbb{N}$ such that for all $(n,p) \in \mathrm{dom}(\phi)$, one has $(n,p) \in \mathrm{dom}(\phi')$ and

$$\begin{aligned} \phi'(n,p) &\in \{\phi(n,p), m_0\}, \\ \|(\alpha \circ \phi)(n,p)\| \geq 3N + 1 &\implies \phi'(n,p) = m_0, \\ \|(\alpha \circ \phi)(n,p)\| \leq 3N &\implies \phi'(n,p) = \phi(n,p). \end{aligned}$$

Note that $\|(\alpha \circ \phi')(n,p)\| \leq 3N + 1$ for all (n,p), and so

$$(\forall \, (n,p) \in \mathrm{dom}(\phi))(\forall \, a \in B(0,N)) \; (\|(\alpha \circ \phi')(n,p) - a\| \leq 4N + 1). \tag{5.7}$$

We shall show that also

$$(\forall \, (n,p) \in \mathrm{dom}(\phi))(\forall \, a \in B(0,N)) \; (\|(\alpha \circ \phi')(n,p) - a\| \leq \|(\alpha \circ \phi)(n,p) - a\|). \tag{5.8}$$

If $\phi'(n,p) = \phi(n,p)$, the claim is trivial. Otherwise, $\phi'(n,p) = m_0$ and $\|(\alpha \circ \phi)(n,p)\| > 3N$; then

$$\|(\alpha \circ \phi)(n,p) - a\| \geq \|(\alpha \circ \phi)(n,p)\| - \|a\| > 2N \geq \|\alpha(m_0) - a\| = \|(\alpha \circ \phi')(n,p) - a\|.$$

By assumption $f(x) \in B(0, N)$ for ν-a.e. $x \in X$; in view of this and using (5.7) and (5.8), we deduce

$$e(f, \delta, \phi'_n, \cdot) \leq \min(e(f, \delta, \phi_n, \cdot), 4N + 1) \qquad \nu\text{-a.e.}$$

This yields

$$\begin{aligned}\int^* e(f, \delta, \phi'_n, \cdot)\, d\nu &\leq \int^*_{[e(f,\delta,\phi'_n,\cdot)>2^{-n}]} (4N + 1) + \int^*_{[e(f,\delta,\phi'_n,\cdot)\leq 2^{-n}]} 2^{-n} \\ &\leq \nu^*([e(f, \delta, \phi_n, \cdot) > 2^{-n}])(4N + 1) + 2^{-n}\nu(X) \\ &\leq 2^{-n}(4N + 1 + \nu(X))\end{aligned}$$

for all $n \in \mathbb{N}$. A suitable Φ is hence given by

$$\Phi(n, p) := \phi'(n + \lceil \log(4N + 1 + M) \rceil, p),$$

where $M \in \mathbb{N}$ is an arbitrary upper bound of $\nu(X)$. Clearly, the map $\phi \mapsto \Phi$ is computable. $\square$

The question whether AE-computability implies MEAN/AE-computability leads to the question whether plain computability implies MEAN-computability. There are patholocial cases in which this is not the case: Consider for example a measure ν on $[0; 1]$ which is not locally finite in 0, a computable constant $c \in \mathbb{R} \setminus \mathbb{Q}$, and the constant function $f(x) := c$ on $[0; 1]$; then it is easy to see that f is $(\rho_{\mathbb{R}}|^{[0;1]}, \rho_{\mathbb{R}})$-computable but not $(\rho_{\mathbb{R}}, \nu_{\mathbb{Q}})^{\nu}_{\mathrm{MEAN}}$-continuous. But if ν is σ-finite (in an effective sense), a reduction can be proved:

Proposition 5.21. *Suppose that there is a sequence $(U_n)_n \in 2^X$ such that the multi-mapping $g : X \rightrightarrows \mathbb{N}$ with*

$$g(x) \ni n \quad :\Longleftrightarrow \quad x \in U_n$$

is well-defined and $(\delta, \nu_{\mathbb{N}})$-computable, and there is a $[\nu_{\mathbb{N}}]^{\omega}$-computable sequence $(M_n)_n$ such that $\nu^*(U_n) \leq M_n$ for every n. Then

(1) $\Lambda(\delta \to \alpha_{\mathrm{Cauchy}})^{\nu} \subseteq \Lambda(\delta \to \alpha)^{\nu}_{\mathrm{MEAN}}$ *and* $[\delta \to \alpha_{\mathrm{Cauchy}}]^{\nu} \leq [\delta \to \alpha]^{\nu}_{\mathrm{MEAN}}$.

(2) $\Lambda(\delta \to \alpha_{\mathrm{Cauchy}})^{\nu}_{\mathrm{AE}} \subseteq \Lambda(\delta \to \alpha)^{\nu}_{\mathrm{MEAN/AE}}$ *and* $[\delta \to \alpha_{\mathrm{Cauchy}}]^{\nu}_{\mathrm{AE}} \leq [\delta \to \alpha]^{\nu}_{\mathrm{MEAN/AE}}$.

Proof. Let a $[\delta \to \alpha_{\mathrm{Cauchy}}]^{\nu}$-name (for item (1)) or a $[\delta \to \alpha_{\mathrm{Cauchy}}]^{\nu}_{\mathrm{AE}}$-name (for item (2)), respectively, of some $[f]_\nu$ be given. This name is an encoded oracle that provides the necessary information to evaluate a mapping $\psi :\subseteq \mathrm{TYPE}(\delta) \to \mathbb{N}^{\omega}$ such that there is a set $A \in S$ with $\mathrm{dom}(\psi) \supseteq \delta^{-1}(A)$ (here $A = X$ for item (1), and A is the complement of a set N with $\nu(N) = 0$ for item (2)) and

$$(x \in A \wedge \delta(p) = x) \implies (\alpha_{\mathrm{Cauchy}} \circ \psi)(p) = f(x) \qquad \text{for } \nu\text{-a.e. } x.$$

In view of the definition of the representation α_{Cauchy}, we can compute a mapping

$$\phi :\subseteq \mathbb{N} \times \mathrm{TYPE}(\delta) \to \mathbb{N}$$

such that $\mathrm{dom}(\phi) \supseteq \mathbb{N} \times \delta^{-1}(A)$ and

$$(\forall\, p \in \delta^{-1}(A))(\forall\, n \in \mathbb{N})\ (d((\alpha \circ \phi_n)(p), (\alpha_{\text{Cauchy}} \circ \psi)(p)) \leq 2^{-n}),$$

and thus

$$x \in A \implies (\forall\, n \in \mathbb{N})\ (e(f, \delta, \phi_n, x) \leq 2^{-n}) \qquad \text{for } \nu\text{-a.e. } x.$$

Let c be a computable realization of g. For every $(n, p) \in \mathbb{N} \times \delta^{-1}(A)$, put

$$\Phi(n, p) := \phi(n + \lceil \log M_{c(p)} \rceil + 1, p).$$

We can clearly compute this Φ on $\mathbb{N} \times \delta^{-1}(A)$. It remains to show that Φ is a $(\delta, \alpha)^{\nu}_{\text{MEAN}}$-realization or $(\delta, \alpha)^{\nu}_{\text{MEAN/AE}}$-realization, respectively, of f. This follows from the estimate

$$\begin{aligned}
\int_A^* e(f, \delta, \Phi_n, x)\, \nu(dx) &\leq \int_A^* \sup_r \chi_{U_r}(x) e(f, \delta, \phi_{n+\lceil \log M_r \rceil + 1}, x)\, \nu(dx) \\
&\leq \int_A^* \sum_{r=0}^{\infty} \chi_{U_r}(x) e(f, \delta, \phi_{n+\lceil \log M_r \rceil + 1}, x)\, \nu(dx) \\
&\leq \sum_{r=0}^{\infty} \int_{A \cap U_r}^* e(f, \delta, \phi_{n+\lceil \log M_r \rceil + 1}, x)\, \nu(dx) \\
&\leq \sum_{r=0}^{\infty} M_r 2^{-(n+\lceil \log M_r \rceil + 1)} \\
&\leq 2^{-n}.
\end{aligned}$$

□

We have seen that APP- and APP/AE-continuity are typically equivalent (see Corollary 5.14). It is an obvious question whether there then is a computable reduction from APP/AE to APP (and hence from AE to APP and from MEAN/AE to APP). We will prove that such a reduction exists under suitable effectivity requirements on the underlying space and measure. To this end, we first exhibit sufficient conditions in the following lemma. In the proof of Theorem 5.23, we then show that these conditions are fulfilled for an important class of spaces and measures.

Lemma 5.22. *Suppose that* $\mathrm{TYPE}(\delta) = \mathbb{N}^{\omega}$ *and*[8] ζ *is a representation of a certain system* $\mathcal{F} \subseteq 2^X$ *of subsets of* X *such that*

(1) the set $\{(x, U) \in X \times \mathcal{F} \ :\ x \in U\}$ *is* $[\delta, \zeta]$*-c.e.; and*

(2) there is a $[\zeta]^{\omega}$*-computable sequence* $(U_r)_r \in \mathcal{F}^{\omega}$ *such that* $X = \bigcup_r U_r$*; and*

[8] With some technical modifications, the lemma can also be proved for $\mathrm{TYPE}(\delta) = \mathbb{N}$.

(3) *from any prefix-free sequence* $(w_\ell)_\ell$ *of elements of* $\mathbb{N}^{<\omega}$ *with*

$$\mu^*\left(X \setminus \bigcup_\ell \delta(w_\ell \mathbb{N}^\omega)\right) = 0 \tag{5.9}$$

and any $r, k \in \mathbb{N}$*, one can* $[\zeta]^\omega$*-compute a sequence* $(V_\ell)_\ell \in \mathcal{F}^\omega$ *and* ζ*-compute a set* $\widetilde{V} \in \mathcal{F}$*, such that*

$$U_r \subseteq \bigcup_\ell V_\ell \cup \widetilde{V}$$

and $\mu^*(L) \leq 2^{-k}$*, where*

$$L := U_r \cap \left(\widetilde{V} \cup \bigcup_\ell (V_\ell \setminus \delta(w_\ell \mathbb{N}^\omega))\right).$$

Then $[\delta \to \alpha]^{\mu^*}_{\mathrm{APP/AE}} \leq [\delta \to \alpha]^{\mu^*}_{\mathrm{APP}}$.

Proof. Let a $[\delta \to \alpha]^{\mu^*}_{\mathrm{APP/AE}}$-name of some $[f]_{\mu^*}$ be given, that is, we are given sufficient information to evaluate a $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$-realization ϕ' of f. Let N be a μ^*-null set with

$$\mathbb{N} \times \delta^{-1}(X \setminus N) \subseteq \mathrm{dom}(\phi')$$

and

$$(\forall\, n \in \mathbb{N})\ \big(\mu^*(\{x \in X \setminus N \ :\ e(f, \delta, \phi'_n, x) > 2^{-n}\}) \leq 2^{-n}\big).$$

We need to demonstrate how to compute a $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-realization ϕ of f. So suppose we are given an input pair $(n, p) \in \mathbb{N} \times \mathrm{dom}(\delta)$. By the standard technique of simulating ϕ' on all possible prefices of all possible inputs, we can compute two double sequences $(w_{m,\ell})_{m,\ell} \in (\mathbb{N}^{<\omega})^{\omega\times\omega}$ and $(a_{m,\ell})_{m,\ell} \in \mathbb{N}^{\omega\times\omega}$ such that the following holds for all m:

(a) The sequence $(w_{m,\ell})_\ell$ is prefix-free; and

(b) $\bigcup_\ell w_{m,\ell}\mathbb{N}^\omega \supseteq \delta^{-1}(X \setminus N)$; and

(c) $\phi'(m, q) = a_{m,\ell}$ whenever $\delta(q) \in X \setminus N$, $q \in w_{m,\ell}\mathbb{N}^\omega$.

For every m, put

$$H_m := \bigcup_\ell \big(\{x \in X \ :\ d(f(x), \alpha(a_{m,\ell})) > 2^{-m}\} \cap \delta(w_{m,\ell}\mathbb{N}^\omega)\big).$$

Using first $\mu^*(N) = 0$ and then property (c), we get

$$\begin{aligned}\mu^*(H_m) &= \mu^*\left(\bigcup_\ell \big(\{x \in X \setminus N \ :\ d(f(x), \alpha(a_{m,\ell})) > 2^{-m}\} \cap \delta(w_{m,\ell}\mathbb{N}^\omega)\big)\right)\\ &\leq \mu^*\big(\{x \in X \setminus N \ :\ e(f, \delta, \phi'_m, x) > 2^{-m}\}\big)\\ &\leq 2^{-m}.\end{aligned}$$

By properties (a) and (b), the sequence $(w_{m,\ell})_\ell$ fulfills the requirements of item (3) for every m, so we can compute sequences $(V_{m,\ell,r,k})_{m,\ell,r,k}$ and $(\widetilde{V}_{m,r,k})_{m,r,k}$ of sets in $\mathcal{F}$ such that

$$(\forall\, m, r, k \in \mathbb{N}) \ \left(U_r \subseteq \bigcup_\ell V_{m,\ell,r,k} \cup \widetilde{V}_{m,r,k} \quad \text{and} \quad \mu^*(L_{m,r,k}) \leq 2^{-k}\right),$$

where

$$L_{m,r,k} := U_r \cap \left(\widetilde{V}_{m,r,k} \cup \bigcup_\ell (V_{m,\ell,r,k} \setminus \delta(w_{m,\ell}\mathbb{N}^\omega))\right).$$

$\phi(n, p)$ is now computed as follows: First find an r_0 such that $\delta(p) \in U_{r_0}$, then put $m_0 := n+1$, $k_0 := n + r_0 + 2$ and find a set

$$A \in \{V_{m_0,\ell,r_0,k_0} \ :\ \ell \in \mathbb{N}\} \cup \{\widetilde{V}_{m_0,r_0,k_0}\}$$

with $\delta(p) \in A$. In case that A is $\widetilde{V}_{m_0,r_0,k_0}$, put out an arbitrary $a \in \mathrm{dom}(\alpha)$; in case that A is V_{m_0,ℓ_0,r_0,k_0} for some ℓ_0, put out a_{m_0,ℓ_0}.

We have to verify that the ϕ computed by this algorithm is correct. Suppose that the input (n, p) has the following properties:

(d) $\delta(p) \notin H_{n+1}$; and

(e) $\delta(p) \notin \bigcup_r L_{n+1,r,n+r+2}$.

Let r_0, m_0, k_0 be the numbers chosen (computed) in the algorithm. Then, by property (e), $\delta(p) \in U_{r_0} \setminus L_{m_0,r_0,k_0}$, which implies

$$\delta(p) \notin \widetilde{V}_{m_0,r_0,k_0} \cup \bigcup_\ell (V_{m_0,\ell,r_0,k_0} \setminus \delta(w_{m_0,\ell}\mathbb{N}^\omega)).$$

Hence, the set A chosen in the algorithm will necessarily have the form V_{m_0,ℓ_0,r_0,k_0}, and furthermore $\delta(p) \in \delta(w_{m_0,\ell_0}\mathbb{N}^\omega)$. Property (d) yields that $\delta(p) \notin H_{m_0}$, so in particular

$$\delta(p) \notin \{x \in X \ :\ d(f(x), \alpha(a_{m_0,\ell_0})) > 2^{-m_0}\} \cap \delta(w_{m_0,\ell_0}\mathbb{N}^\omega),$$

which implies

$$\delta(p) \notin \{x \in X \ :\ d(f(x), \alpha(a_{m_0,\ell_0})) > 2^{-m_0}\}.$$

As $\phi(n,p) = a_{m_0,\ell_0}$, we arrive at

$$d((f\circ\delta)(p),(\alpha\circ\phi)(n,p)) \leq 2^{-m_0} < 2^{-n}.$$

It remains to estimate the content of the set of $\delta(p)$'s that do not fulfill (d) and (e):

$$\mu^*([e(f,\delta,\phi_n,\cdot) > 2^{-n}]) \leq \mu^*(H_{n+1}) + \sum_{r=0}^{\infty}\mu^*(L_{n+1,r,n+r+2}) \leq 2^{-(n+1)} + \sum_{r=0}^{\infty} 2^{-(n+r+2)} = 2^{-n}.$$

□

Theorem 5.23. *Let (X,ϑ) be a computably regular computable T_0-space. Let ν be a Borel measure on X such that there is a $\vartheta_{\mathcal{O}<}$-computable sequence $(U_r)_r \in (\tau_\vartheta)^\omega$ such that $X = \bigcup_r U_r$ and the sequence $(\nu_r)_r$ of measures with*

$$\nu_r(A) = \nu(A\cap U_r), \qquad A\in\mathcal{B}(X),\ r\in\mathbb{N},$$

is in $(\mathcal{M}_0)^\omega$ and $[\vartheta_{\mathcal{M}_0<}]^\omega$-computable. Then $[\vartheta_{\mathrm{std}} \to \alpha]^\nu_{\mathrm{APP/AE}} \leq [\vartheta_{\mathrm{std}} \to \alpha]^\nu_{\mathrm{APP}}$.

Proof. It is sufficient to check that conditions (1), (2), (3) of Lemma 5.22 are fulfilled for $\delta = \vartheta_{\mathrm{std}}$, $\mu^* = \nu^*$, $\mathcal{F} = \tau_\vartheta$, $\zeta = \vartheta_{\mathcal{O}<}$, and $(U_r)_r$ as in the statement of this theorem. Conditions (1) and (2) are clearly fulfilled. We check condition (3): So let a sequence $(w_\ell)_\ell$ and numbers r,k as in condition (3) be given. Lemma 2.9 yields that we can $[\vartheta_{\mathcal{O}<}]^\omega$-compute the sequence $(V_\ell)_\ell$ with $V_\ell := \vartheta_{\mathrm{std}}(w_\ell\mathbb{N}^\omega)$ for every $\ell\in\mathbb{N}$, and so we can $\vartheta_{\mathcal{O}<}$-compute the set $U_r\cap\bigcup_\ell V_\ell$. By the definition of $\vartheta_{\mathcal{O}<}$, this means that we can $[\vartheta]^\omega$-compute a sequence $(W_n)_n$ with $\bigcup_n W_n = U_r\cap\bigcup_\ell V_\ell$. Compute reg on each set $W_0, W_1, \ldots$; let $(\widetilde{W}_{0,m}, A_{0,m})_m, (\widetilde{W}_{1,m}, A_{1,m})_m, \ldots$ be the results. Then

$$(\forall\, n\in\mathbb{N})\ \big(W_n = \bigcup_m \widetilde{W}_{n,m}\big), \tag{5.10}$$

and

$$(\forall\, n,m\in\mathbb{N})\ (W_{n,m}\subseteq A_{n,m}\subseteq W_n). \tag{5.11}$$

Equation (5.10) yields $U_r\cap\bigcup_\ell V_\ell = \bigcup_s \widetilde{V}_s$, where $\widetilde{V}_s := \bigcup_{n,m=0}^{s}\widetilde{W}_{n,m}$. We can $[\vartheta_{\mathcal{O}<}]^\omega$-compute $(\widetilde{V}_s)_s$, and thus we can $[\rho_{\mathbb{R}<}]^\omega$-compute $(\nu(\widetilde{V}_s))_s$. By assumption (5.9), the latter sequence converges monotonously to the number $\nu(U_r)$, which we can $\rho_{\mathbb{R}>}$-compute. We can hence find a number s_0 with

$$\nu\big(U_r\setminus\bigcup_{n,m=0}^{s_0}\widetilde{W}_{n,m}\big) < 2^{-k}. \tag{5.12}$$

Consider the set $A := \bigcup_{n,m=0}^{s_0} A_{m,n}$, which we can $\vartheta_{\mathcal{C}>}$-compute. Combining (5.12) and the first inclusion in (5.11) yields $\nu(U_r\setminus A) < 2^{-k}$. The second inclusion in (5.11) yields $A\subseteq U_r\cap\bigcup_\ell V_\ell$. Choose $\widetilde{V} = X\setminus A$. We have

$$2^{-k} > \nu(U_r\setminus A) = \nu(U_r\cap\widetilde{V}) = \nu\left(U_r\cap\left(\widetilde{V}\cup\bigcup_\ell(V_\ell\setminus\delta(w_\ell\mathbb{N}^\omega))\right)\right)$$

and $U_r\subseteq\bigcup_\ell V_\ell\cup\widetilde{V}$ as required. □

Theorem 5.23 is in fact a generalization of a result of Parker (cf. [Par03, Theorem II]), who proves that the characteristic function of a subset of Euclidean space is APP-computable if it is AE-computable with respect to Lebesgue measure. Parker's proof already contains the central ideas of our proof of Theorem 5.23.

5.4.2 Counter-examples

We shall give strong counter-examples – i.e. examples involving functions from $[0;1]$ to $\mathbb{R}$ and the Lebesgue measure – for the continuous reducibilities that have not been treated in the previous subsection.

Proposition 5.24. *There is a set $S \subseteq [0;1]$ such that χ_S is $(\rho_{\mathbb{R}}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN}}$-computable but not $(\rho_{\mathbb{R}}, \rho_{\mathbb{R}})^{\lambda}_{\mathrm{AE}}$-continuous.*

Proof. Parker (see [Par03, Theorem IV]) considers a positive-measure Cantor set $S \subseteq [0;1]$ and proves that χ_S is $(\rho_{\mathbb{R}}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{APP}}$-computable but not $(\rho_{\mathbb{R}}, \rho_{\mathbb{R}})^{\lambda}_{\mathrm{AE}}$-continuous (although he does not use these terms). By Proposition 5.20, χ_S is even $(\rho_{\mathbb{R}}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN}}$-computable. □

Proposition 5.25. *(1) There exists a function $f : [0;1] \to \mathbb{R}$ which is $(\rho_{\mathbb{R}}|^{[0;1]}, \rho_{\mathbb{R}})^{\lambda}_{\mathrm{AE}}$- and $(\rho_{\mathbb{R}}|^{[0;1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN/AE}}$-computable but not $(\rho_{\mathbb{R}}|^{[0;1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN}}$-continuous.*

(2) There exists a function $f : [0;1] \to \mathbb{R}$ which is $(\rho_{\mathbb{R}}|^{[0;1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{APP}}$-computable but not $(\rho_{\mathbb{R}}|^{[0;1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN/AE}}$-continuous.

Proof. Recall that $\rho_{\mathbb{R}}|^{[0;1]}$ is an admissible representation of $[0;1]$. We can hence apply Corollary 5.18 and have that any $(\rho_{\mathbb{R}}|^{[0;1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN}}$-continuous function is locally λ-integrable

For item (1), simply consider $f(x) := x^{-1} \cdot \chi_{(0;1]}(x)$, which clearly is computable and MEAN-computable on $(0;1]$, but not locally integrable in 0.

For item (2), we need a more elaborate example: For every $a \in [0;1], n \in \mathbb{N}$, define

$$f_{a,n}(x) := (x-a)^{-1}\chi_{(a;a+2^{-n}]\cap[0;1]}(x).$$

Let $(a_n)_{n\in\mathbb{N}}$ be a computable dense sequence of rationals in $[0;1]$. Choose $\widetilde{f} := \sup_{n\in\mathbb{N}} f_{a_n,n}$. $\widetilde{f}$ is a measurable function into $\overline{\mathbb{R}}$, that is not integrable on any open subset of $[0;1]$, because any such open subset must contain an interval of the form $[a_n, a_n + \varepsilon] =: I$ and one already has $\int_I f_{a_n,n}\, d\lambda = \infty$. Obviously, $\widetilde{f}(x) = \infty$ implies that x is contained in infinitely many of the $(a, a+2^{-n}]$, and hence Cantelli's Theorem yields $\lambda([\widetilde{f} = \infty]) = 0$. So, the function $f := \widetilde{f} \cdot \chi_{[\widetilde{f}\neq\infty]}$ is into $\mathbb{R}$ and is still measurable and nowhere integrable. $f|_{X\setminus N}$ is still nowhere

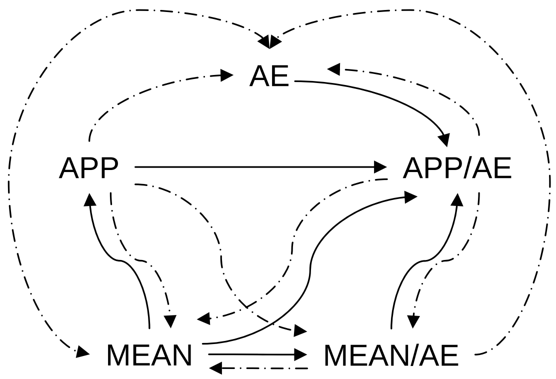


Figure 5.1: The graphic represents the transitive closure of the implications and non-implications proved in Proposition 5.19 and Subsection 5.4.2. A solid arrow indicates computable reduction, a dashed arrow indicates a strong counter-example.

integrable for any ν-null set N. So f is not $(\rho_{\mathbb{R}}|^{[0;1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{MEAN/AE}}$-continuous. It remains to show that f is $(\rho_{\mathbb{R}}|^{[0;1]}, \nu_{\mathbb{Q}})^{\lambda}_{\mathrm{APP}}$-computable. For every n, put $f_n := \sup_{k \leq n} f_{a_k,k}$ and note that

$$\lambda([f \neq f_n]) \leq \lambda\Big(\bigcup_{k>n}(a_k; a_k + 2^{-n}]\Big) \leq \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n}.$$

Using this, it is sufficient to show that $(f_n)_n$ is $[[\rho_{\mathbb{R}}|^{[0;1]} \to \nu_{\mathbb{Q}}]^{\lambda}_{\mathrm{APP}}]^{\omega}$-computable. (In order to compute f with error level 2^{-n}, compute f_{n+1} with error level $2^{-(n+1)}$.) It is easy to see that $(f_n)_n$ is $[[\rho_{\mathbb{R}}|^{[0;1]} \to \rho_{\mathbb{R}}]^{\lambda}_{\mathrm{AE}}]^{\omega}$-computable. The claim then follows from Proposition 5.19.2 and Theorem 5.23. $\square$

5.5 Computability of vector-valued integration

5.5.1 The Pettis integral

Recall that the cylindrical σ-algebra $\mathcal{E}(Z)$ on a topological vector space Z over $\mathbb{F}$ is the smallest σ-algebra $\mathcal{A}$ on Z such that every $f \in X^*$ is $(\mathcal{A}, \mathcal{B}(\mathbb{F}))$-measurable.

We collect a number of definitions and basic facts from [VTC87, Section II.3.1]: Suppose that Y is a normed space over $\mathbb{F}$ and that $f : X \to Y$ is an $(\mathcal{S}, \mathcal{E}(Y))$-measurable mapping such that

$$(\forall\, g \in Y^*)\ \Big(\int |g \circ f|\, d\nu < \infty\Big). \tag{5.13}$$

Then we call an element y_f of Y **(Pettis) integral** of f with respect to ν if

$$(\forall\, g \in Y^*)\ \Big(\int g \circ f\, d\nu = g(y_f)\Big).$$

If there is an integral of f, then it is unique and we denote it by $\int f\, d\nu$. The mappings for which the integral exists form a vector space on which $f \mapsto \int f\, d\nu$ is linear. For real-valued f, the Pettis integral is equal to the usual integral.

Suppose that $f : X \to Y$ is $(\mathcal{S}, \mathcal{B}(Y))$-measurable. If

$$\int \|f\|\, d\nu < \infty, \tag{5.14}$$

then also (5.13) is fulfilled. If $\int f\, d\nu$ exists, then

$$\Big\| \int f\, d\nu \Big\| \le \int \|f\|\, d\nu.$$

(We will use this estimate frequently.) To ensure the existence of $\int f\, d\nu$, it is sufficient that (5.14) holds and Y is complete.

If f has the form

$$f(x) = \sum_{i=1}^{n} y_i \chi_{A_i}(x)$$

with $y_1, \dots, y_n \in Y$ and disjoint subsets $A_1, \dots, A_n$ of X, it is easy to see that

$$\int f\, d\nu = \sum_{i=1}^{n} y_i \nu(A_i).$$

As for usual integrals (cf. Subsection 5.1.1), we shall use the notational convention

$$\int f\, d\nu := \int f\, d\overline{\nu},$$

if f is merely $(\mathcal{S}_\nu, \mathcal{E}(Y))$-measurable.

5.5.2 Effective integration of MEAN-continuous functions

Under what circumstances and for what representations is $(f, \nu) \mapsto \int f\, d\nu$ computable? The next theorem gives examples for the special case of MEAN-continuous functions on computable T_0-spaces. The corresponding integration algorithms will be uniform in both the mapping and the measure.

Before we state the theorem, we have to make a technical

Remark 5.26. Suppose that (X, ϑ) is a computable T_0-space, Y is a normed space over $\mathbb{F}$, $\nu \in \mathcal{M}_0(X)$, and $[f]_\nu \in \Lambda(\vartheta_{\mathrm{std}} \to \alpha)^\nu_{\mathrm{MEAN}}$. As ν is finite, we have that $\mathrm{MEAS}_{\nu^*} = \mathcal{B}(X)_\nu$ (see [Coh80, Exercise 1.5.9]). As ϑ_{std} is open (cf. Lemma 2.9), one has $\sigma(\vartheta_{\mathrm{std}}^{-1}) \subseteq \mathcal{B}(X) \subseteq \mathcal{B}(X)_\nu$. Then Corollary 5.15 implies that f is $(\mathcal{B}(X)_\nu, \mathcal{B}(Y))$-measurable. It is clear that the existence and (in case of existence) the value of $\int f\, d\nu$ only depend on the equivalence class $[f]_\nu$ of f.

Theorem 5.27. *Suppose that (X, ϑ) is a computable T_0-space and Y is a normed space over $\mathbb{F}$ such that norm, vector addition, and scalar multiplication are computable when Y is represented by α_{Cauchy} and $\mathbb{F}$ is represented by $\rho_{\mathbb{F}}$. Put*

$$L := \{(\nu, [f]_\nu) \ : \ \nu \in \mathcal{M}_0(X),\ [f]_\nu \in \Lambda(\vartheta_{\mathrm{std}} \to \alpha)^\nu_{\mathrm{MEAN}},\ \int f\, d\nu \textit{ exists}\}.$$

Let Ξ be the representation of L defined ad-hoc by

$$\Xi\langle p, q\rangle_{**} = (\nu, f) \quad :\Longleftrightarrow \quad \vartheta_{\mathcal{M}_0}(p) = \nu \quad \textit{and} \quad [\vartheta_{\mathrm{std}} \to \alpha]^\nu_{\mathrm{MEAN}}(q) = [f]_\nu.$$

(1) $((\nu, [f]_\nu), K) \mapsto \int f\, d\nu$ for $(\nu, [f]_\nu) \in L$ and $K \in \tau^\kappa_\vartheta$ such that f vanishes ν-a.e. outside K is $([\Xi, \vartheta_{\mathcal{K}>}], \alpha_{\mathrm{Cauchy}})$-computable.

(2) $((\nu, [f]_\nu), b) \mapsto \int f\, d\nu$ for $(\nu, [f]_\nu) \in L$ and $b \in \mathbb{N}$ such that $\|f\| \leq b$ ν-a.e. is $([\Xi, \nu_{\mathbb{N}}], \alpha_{\mathrm{Cauchy}})$-computable.

(3) $((\nu, [f]_\nu), c) \mapsto \int f\, d\nu$ for $(\nu, [f]_\nu) \in L$ and $c = \int \|f\|\, d\nu$ is $([\Xi, \rho_{\mathbb{R}>}], \alpha_{\mathrm{Cauchy}})$-computable.

Proof. The proofs of items (1), (2), and (3) start the same: Let $(\nu, [f]_\nu)$ be the Ξ-encoded input; so we are in particular given an $(\vartheta_{\mathrm{std}}, \alpha)^\nu_{\mathrm{MEAN}}$-realization Φ of f. It is sufficient to demonstrate how to α_{Cauchy}-compute a 2^{-k}-approximation to $\int f\, d\nu$ for any given $k \in \mathbb{N}$. So fix an arbitrary k (it will be clear that the construction is uniform in k). By simulation of Φ_{k+2}, we can compute a prefix-free sequences $(w_\ell)_\ell \in (\mathbb{N}^{<\omega})^\omega$ and a sequence $(a_\ell)_\ell \in \mathbb{N}^\omega$ such that $\mathrm{dom}(\vartheta_{\mathrm{std}}) \subseteq \bigcup_\ell w_\ell \mathbb{N}^\omega$ – and hence

$$X = \bigcup_\ell \vartheta_{\mathrm{std}}(w_\ell \mathbb{N}^\omega)$$

– and such that Φ_{k+1} is constantly equal to a_ℓ on $w_\ell \mathbb{N}^\omega \cap \mathrm{dom}(\vartheta_{\mathrm{std}})$ for every $\ell \in \mathbb{N}$. By Lemma 2.9, we can $[\vartheta_{\mathcal{O}<}]^\omega$-compute the sequence $(V_\ell)_\ell$ with $V_\ell := \vartheta_{\mathrm{std}}(w_\ell \mathbb{N}^\omega)$ for every ℓ. Note that

$$(\forall\, x \in X)\ \big(e(f, \delta, \Phi_{k+2}, x) = \sup_\ell \chi_{V_\ell}(x) \|f(x) - \alpha(a_\ell)\|\big). \tag{5.15}$$

It follows from the definition of $\vartheta_{\mathcal{M}_0}$ that we can $[\vartheta_{\mathcal{O}<}]^\omega$-compute a sequence $(W_n)_n$ such that

$$(\forall\, \ell \in \mathbb{N})\ \big(V_\ell = \bigcup_m W_{\langle \ell, m\rangle_{**}}\big)$$

and such that we can compute ν on algebraic expressions of the W_n. Hence, if we put

$$A_n := W_n \setminus \bigcup_{i<n} W_i$$

for every n, we can $[\rho_{\mathbb{R}}]^\omega$-compute the sequence $(\nu(A_n))_n$. Define $v_{\langle \ell,m\rangle_{**}} := \alpha(a_\ell)$ for all $\ell, m \in \mathbb{N}$. Then define $s : X \to Y$ by

$$s(x) := \sum_{n=0}^{\infty} \chi_{A_n}(x) \cdot v_n.$$

The definitions of the A_n and V_n and equation (5.15) yield

$$\begin{aligned}
\int \|f - s\|\, d\nu &= \int \Big\| f(x) - \sum_{n=0}^{\infty} \chi_{A_n}(x) \cdot v_n \Big\| \,\nu(dx) \\
&= \int \sup_n \chi_{A_n}(x) \|f(x) - v_n\| \,\nu(dx) \\
&\leq \int \sup_n \chi_{W_n}(x) \cdot \|f(x) - v_n\| \,\nu(dx) \\
&= \int \sup_\ell \chi_{V_\ell}(x) \cdot \|f(x) - \alpha(a_\ell)\| \,\nu(dx) \\
&= \int^* e(f, \delta, \Phi_{k+2}, x)\, \nu(dx) \\
&\leq 2^{-(k+2)}.
\end{aligned} \tag{5.16}$$

For every $n \in \mathbb{N}$, put

$$B_n := \bigcup_{i \leq n} A_i = \bigcup_{i \leq n} W_i,$$

$$y_n := \sum_{i=0}^{n} \nu(A_i) v_i,$$

$$s_n(x) := \chi_{B_n}(x) \cdot s(x) = \sum_{i=0}^{n} \chi_{A_i}(x) \cdot v_i.$$

One immediately verifies that the sequence $(y_n)_n$ can be $[\alpha_{\mathrm{Cauchy}}]^\omega$-computed, and that $\int s_n\, d\nu = y_n$ for every n. Combining this with (5.16) yields for every n:

$$\begin{aligned}
\Big\| \int f\, d\nu - y_n \Big\| \leq \int \|f - s_n\|\, d\nu &= \int \chi_{X \setminus B_n} \cdot \|f\|\, d\nu + \int \chi_{B_n} \cdot \|f - s\|\, d\nu \\
&\leq \int \chi_{X \setminus B_n} \cdot \|f\|\, d\nu + 2^{-(k+2)}.
\end{aligned}$$

So it is sufficient to compute an n such that

$$\int \chi_{X \setminus B_n} \cdot \|f\|\, d\nu \leq 2^{-(k+1)} + 2^{-(k+2)}. \tag{5.17}$$

For item (1): Let K be the $\vartheta_{\mathcal{K}>}$-encoded additional input. We can $[\vartheta_{\mathcal{O}<}]^\omega$-compute the sequence $(B_n)_n$. As $(B_n)_n$ is ascending and covers X, we can effectively choose n such that $K \subseteq B_n$. By the assumption on K, we have that f vanishes ν-a.e. outside B_n. Thus the left-hand side of (5.17) is equal to zero.
For item (2): Recall that $\vartheta_{\mathcal{M}_0} \le \vartheta_{\mathcal{M}_0<}$, and so we can $[\rho_{\mathbb{R}<}]^\omega$-compute the sequence $(\nu(B_n))_n$. As we can also ρ-compute $\nu(X)$, we can effectively find an n such that

$$\nu(X \setminus B_n) \le b^{-1}(2^{-(k+1)} + 2^{-(k+2)}).$$

For this n, (5.17) is fulfilled.
For item (3): From (5.16) and by Monotone Convergence, it follows that

$$2^{-(k+2)} \ge \int \|f\|\, d\nu - \int \|s\|\, d\nu = \int \|f\|\, d\nu - \lim_{n\to\infty} \int \chi_{B_n} \cdot \|s\|\, d\nu.$$

The sequence under the limit on the right-hand side can be $[\rho_{\mathbb{R}}]^\omega$-computed because

$$\int \chi_{B_n} \cdot \|s\|\, d\nu = \int \|s_n\|\, d\nu = \sum_{i=0}^{n} \nu(A_i)\|v_i\|$$

for every n. By assumption, we are given a $\rho_{\mathbb{R}>}$-name of $\int \|f\|\, d\nu$, so we can effectively find an n such that

$$2^{-(k+1)} \ge \int \|f\|\, d\nu - \int \|s_n\|\, d\nu.$$

This estimate and (5.16) finally yield

$$\begin{aligned}
&\int \chi_{X\setminus B_n} \cdot \|f\|\, d\nu \\
&= \Big(\int \|f\|\, d\nu - \int \chi_{B_n} \cdot \|s\|\, d\nu\Big) + \Big(\int \chi_{B_n} \cdot \|s\|\, d\nu - \int \chi_{B_n} \cdot \|f\|\, d\nu\Big) \\
&\le 2^{-(k+1)} + 2^{-(k+2)}.
\end{aligned}$$

□

In view of Theorem 2.29, we have the following corollary:

Corollary 5.28. *If the space (X, ϑ) in the previous theorem is computably regular, then the theorem still holds true if the definition of Ξ is changed in the following way:*

$$\Xi\langle p, q\rangle_{**} = (\nu, [f]_\nu) \quad :\Longleftrightarrow \quad \vartheta_{\mathcal{M}_0<}(p) = \nu \quad \textit{and} \quad [\vartheta_{\mathrm{std}} \to \alpha]^{\nu}_{\mathrm{MEAN}}(q) = [f]_\nu.$$

□

5.6 Composition of probabilistically computable mappings

We will now prove two theorems on APP-computability of compositions of mappings. The first result is a partial answer to the natural question whether the composition of two APP-computable mappings is still APP-computable. We will see the APP/AE-concept arise naturally. The second result is (a uniform version of) the observation that APP-computability is preserved under composition with computable mappings with a computable modulus of uniform continuity.

Theorem 5.29. *Let (Z, d', α') be another computable metric space. Let $f : X \to Y$ and $g : Y \to Z$ be mappings. If f is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-computable and g is $(\alpha_{\mathrm{Cauchy}}, \alpha')^{\mu^* \circ f^{-1}}_{\mathrm{APP}}$-computable, then $g \circ f$ is $(\delta, \alpha')^{\mu^*}_{\mathrm{APP/AE}}$-computable.*

Proof. Let ϕ be a $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-realization of f. Consider the mapping

$$a :\subseteq \mathbb{N} \times \mathbb{N}^\omega \to \mathbb{N}^\omega, \quad a(n, p) := (\phi(n + k + 1, p))_k.$$

For every $p \in \mathrm{dom}(\delta)$ with

$$(\forall\, k \in \mathbb{N})\; (d((\alpha \circ \phi)(n + k + 1, p), (f \circ \delta)(p)) \leq 2^{-(k+1)}),$$

we have that $a(n, p)$ is a Cauchy name of $(f \circ \delta)(p)$. This observation, in connection with the definition of the local error, yields that, for every n, the set

$$R_n := \{x \in X \;:\; (\exists\, p \in \delta^{-1}\{x\})\; (a(n, p) \notin \alpha^{-1}_{\mathrm{Cauchy}}\{f(x)\})\}$$

is contained in the set

$$\bigcup_k [e(f, \delta, \phi_{n+k+1}, \cdot) > 2^{-(k+1)}];$$

our assumption on ϕ then yields

$$\begin{aligned} \mu^*(R_n) &\leq \mu^*\big(\bigcup_k [e(f, \delta, \phi_{n+k+1}, \cdot) > 2^{-(k+1)}]\big) \\ &\leq \sum_{k=0}^{\infty} \mu^*([e(f, \delta, \phi_{n+k+1}, \cdot) > 2^{-(n+k+1)}]) \\ &\leq \sum_{k=0}^{\infty} 2^{-(n+k+1)} = 2^{-n}. \end{aligned} \tag{5.18}$$

Now let ϕ' be a $(\alpha_{\mathrm{Cauchy}}, \alpha')^{\mu^* \circ f^{-1}}_{\mathrm{APP}}$-realization of g. Consider the mapping $\widetilde{\phi} :\subseteq \mathbb{N} \times \mathbb{N}^\omega \to \mathbb{N}$, computed by the following procedure: “On input $(n, p) \in \mathbb{N} \times \mathbb{N}^\omega$, run a dovetailed process that simulates the computation of a machine for ϕ' on all inputs $(n+1, a(n+m+2, p))$, $m \geq 0$. As soon as the first such simulation halts, put out its output and halt.” $\widetilde{\phi}$ is surely defined on every

(n,p) with $p \in \mathrm{dom}(\delta)$ and $\delta(p) \notin \bigcap_m R_{n+m+2}$, because in this case there surely is an m such that $a(n+m+2,p) \in \mathrm{dom}(\alpha_{\text{Cauchy}})$. Hence, if we put

$$N := \bigcup_n \bigcap_m R_{n+m+2},$$

then $\mathbb{N} \times \mathrm{dom}(\delta|^{X\setminus N}) \subseteq \mathrm{dom}(\widetilde{\phi})$. It follows easily from (5.18) that $\mu^*(N) = 0$. In order to show that $\widetilde{\phi}$ is a $(\delta|^{X\setminus N}, \alpha')^{\mu^* \circ f^{-1}}_{\text{APP}}$-realization of $g \circ f|_{X\setminus N}$, it remains to prove

$$(\forall\, n \in \mathbb{N})\ \big((\mu^* \circ f^{-1})(\{x \in X \setminus N\ :\ e(g \circ f, \delta, \widetilde{\phi}_n, x) > 2^{-n}\}) \leq 2^{-n}\big). \tag{5.19}$$

If for some $n \in \mathbb{N}$, $x \in X$, we have that both the conditions

$$(\forall\, p \in \delta^{-1}\{x\})(\forall\, m \in \mathbb{N})\ \big(a(n+m+2,p) \in \alpha^{-1}_{\text{Cauchy}}\{f(x)\}\big)$$

and

$$(\forall\, q \in \alpha^{-1}_{\text{Cauchy}}\{f(x)\})\ \big(d'((\alpha' \circ \phi')(n+1,q), (g \circ f)(x)) \leq 2^{-(n+1)}\big)$$

are fulfilled, then it follows from the construction of our procedure for $\widetilde{\phi}$ that

$$(\forall\, p \in \delta^{-1}\{x\})\ \big(d'((\alpha' \circ \widetilde{\phi})(n,p), (g \circ f)(x)) \leq 2^{-(n+1)} \leq 2^{-n}\big).$$

By the definition of the local error, this implies

$$\begin{aligned}\{x \in X \setminus N\ :\ e(g \circ f, \delta, \widetilde{\phi}_n, x) > 2^{-n}\}& \\ \subseteq \bigcup_m R_{n+m+2} \cup \{x \in X\ :\ e(g, \alpha_{\text{Cauchy}}, \phi'_{n+1}, f(x)) > 2^{-(n+1)}\}&.\end{aligned}$$

Condition (5.19) is hence fulfilled because

$$\mu^*\left(\bigcup_m R_{n+m+2}\right) \leq 2^{-(n+1)}$$

by (5.18), and

$$(\mu^* \circ f^{-1})[e(g, \alpha_{\text{Cauchy}}, \phi'_{n+1}, \cdot) > 2^{-(n+1)}] \leq 2^{-(n+1)}$$

by assumption. $\square$

Proposition 5.30. *Let (Z, d', α') be another computable metric space. The mapping*

$$([f]_{\mu^*}, g) \mapsto [g \circ f]_{\mu^*}$$

for $f \in \Lambda(\delta \to \alpha)^{\mu^*}_{\text{APP}}$ and total $g \in C(Y,Z)_{\text{uni}}$ is

$$([[\delta \to \alpha]^{\mu^*}_{\text{APP}}, [\alpha_{\text{Cauchy}} \to \alpha'_{\text{Cauchy}}]_{\text{uni}}], [\delta \to \alpha']^{\mu^*}_{\text{APP}})\textit{-computable}.$$

Proof. Let ϕ be the given $(\delta,\alpha)^{\mu^*}_{\mathrm{APP}}$-realization of f, and let $m:\mathbb{N}\to\mathbb{N}$ be the given modulus of uniform continuity of g. A $(\delta,\alpha')^{\mu^*}_{\mathrm{APP}}$-realization ϕ' of $g\circ f$ can be computed as follows: On input $(n,p)\in\mathbb{N}\times\mathrm{dom}(\delta)$, compute an $a\in\mathbb{N}$ such that

$$d'(\alpha(a),(g\circ\phi)(\max\{n,m(n+1)\},p))<2^{-(n+1)}$$

and put it out. In order to see that this procedure is correct, note that for all $n\in\mathbb{N}$ and $p\in\mathrm{dom}(\delta)$

$$\begin{aligned}&d((f\circ\delta)(p),\phi(\max\{n,m(n+1)\},p))\le 2^{-m(n+1)}\\ &\implies d'((g\circ f\circ\delta)(p),(g\circ\phi)(\max\{n,m(n+1)\},p))\le 2^{-(n+1)}\\ &\implies d'((\alpha\circ\phi')(n,p),(g\circ f\circ\delta)(p))\le 2^{-n}.\end{aligned}$$

This implies that for every n

$$\begin{aligned}&[e(g\circ f,\delta,\phi',\cdot)>2^{-n}]\\ &\subseteq[e(f,\delta,\phi_{\max\{n,m(n+1)\}},\cdot))>2^{-m(n+1)}]\\ &\subseteq[e(f,\delta,\phi_{\max\{n,m(n+1)\}},\cdot))>2^{-\max\{n,m(n+1)\}}].\end{aligned}$$

It remains to note that, by assumption, the set on the right-hand side has μ^*-content at most $2^{-\max\{n,m(n+1)\}}\le 2^{-n}$. $\square$

The next result is on the computability of measures induced by APP-computable mappings. We have to make a technical remark similar to the one preceding Theorem 5.27:

Remark 5.31. Suppose that (X,ϑ) is a computable T_0-space, $\nu\in\mathcal{M}_0(X)$, and $[f]_\nu\in\Lambda(\vartheta_{\mathrm{std}}\to\alpha)^{\nu}_{\mathrm{APP}}$. We have that f is $(\mathcal{B}(X)_\nu,\mathcal{B}(Y))$-measurable, which can be seen similarly as in Remark 5.26 (just invoke Proposition 5.12 instead of Corollary 5.15). It is clear that the image measure $\nu\circ f^{-1}$ only depends on the equivalence class $[f]_\nu$ of f.

Theorem 5.32. *Let (X,ϑ) be a computable T_0-space. Put*

$$L:=\{(\nu,[f]_\nu)\ :\ \nu\in\mathcal{M}_0(X),\ [f]_\nu\in\Lambda(\vartheta_{\mathrm{std}}\to\alpha)^{\nu}_{\mathrm{APP}}\}.$$

Let Ξ be the representation of L ad-hoc defined by

$$\Xi\langle p,q\rangle_{**}=(\nu,f)\quad:\Longleftrightarrow\quad\vartheta_{\mathcal{M}_0}(p)=\nu\quad\textit{and}\quad[\vartheta_{\mathrm{std}}\to\alpha]^{\nu}_{\mathrm{APP}}(q)=[f]_\nu.$$

The mapping $(\nu,[f]_\nu)\mapsto\nu\circ f^{-1}$ for $(\nu,[f]_\nu)\in L$ is $(\Xi,\vartheta^{\alpha}_{\mathcal{M}_0<})$-computable.

Proof. It follows from the definition of $\vartheta^{\alpha}_{\mathcal{M}_0<}$ that it is sufficient to demonstrate how to $\rho_<$-compute the $\nu\circ f^{-1}$-content of any open set

$$V=\bigcup_{i=1}^{m}B(x_i,r_i)$$

given an $[\alpha, \nu_{\mathbb{Q}}]^{<\omega}$-name of $(x_i, r_i)_{i=1}^m \in (\mathcal{R}_\alpha \times (\mathbb{Q} \cap (0; \infty)))^{<\omega}$, a $\vartheta_{\mathcal{M}_0}$-name of ν and a $[\vartheta_{\text{std}} \to \alpha]^\nu_{\text{APP}}$-name of f. It is easy to see that, for every $i \in \{1, \dots, m\}$ and $n \in \mathbb{N}$, we can $[\alpha_{\text{Cauchy}} \to \rho_{\mathbb{R}}]_{\text{uni}}$-compute the function

$$g_{i,n}(x) := \max\{0, \min\{1, 2^n(r_i - d(x_i, x))\}\}.$$

The sequence $(g_{i,n})_n$ converges pointwise monotonously increasing to the characteristic function of $B(x_i, r_i)$. For every $n \in \mathbb{N}$, we can $[\alpha_{\text{Cauchy}} \to \rho_{\mathbb{R}}]_{\text{uni}}$-compute the function

$$g_n(x) := \max_{1 \le i \le m} g_{i,n}(x).$$

We have $0 \le g_n \le 1$ and[9] $g_n \nearrow \chi_V$, and hence by Monotone Convergence

$$\int g_n \circ f \, d\nu = \int g_n \, d(\nu \circ f^{-1}) \nearrow \int \chi_V \, d(\nu \circ f^{-1}) = (\nu \circ f^{-1})(V).$$

It is hence sufficient to demonstrate how to $[\rho_{\mathbb{R}}]^\omega$-compute the sequence $(\int g_n \circ f \, d\nu)_n$. It follows from Proposition 5.30 that we can $[[\vartheta_{\text{std}} \to \nu_{\mathbb{Q}}]^\nu_{\text{APP}}]^\omega$-compute the sequence $(g_n \circ f)_n$. The sequence is uniformly bounded by 1, so we can even $[[\vartheta_{\text{std}} \to \nu_{\mathbb{Q}}]^\nu_{\text{MEAN}}]^\omega$-compute it by Proposition 5.20. The corresponding sequence of integrals can now be computed by Theorem 5.27.2. $\square$

[9] $\nearrow$ denotes the relation "converges pointwise monotonously from below to".

Chapter 6

Computability and Gaussian Measures

6.1 Preliminaries on Gaussian measures

6.1.1 Definition and basic facts

We recall the definition of Gaussian measures, and we also collect a number of facts that will be needed below. Our main reference is [Bog98].

A Borel probability measure γ on $\mathbb{R}$ is called **Gaussian** if there is an $a \in \mathbb{R}$ such that γ is the Dirac measure δ_a concentrated in a (i.e. $\delta_a(A) = \chi_A(a)$ for all $A \in \mathcal{B}(\mathbb{R})$) or γ has a density (with respect to λ) of the form

$$t \mapsto \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-a)^2}{2\sigma^2}\right)$$

for some $\sigma > 0$. It is well-known that in this case $a = \int x\,\gamma(dx)$ and $\sigma^2 = \int (x-a)^2\,\gamma(dx)$. If $a = 0$ and $\sigma = 1$, one speaks of the **standard Gaussian measure**.

This definition extends to measures on **locally convex spaces**. Recall that a topological vector space is locally convex if it is Hausdorff and every neighborhood of every element of the space contains a convex neighborhood of that element (see [Sch99]). So let us fix a locally convex space X over $\mathbb{R}$.

A measure γ defined on the cylindrical σ-algebra $\mathcal{E}(X)$ is called **Gaussian** if all measures of the form $\gamma \circ f^{-1}$ with $f \in X^*$ are Gaussian on $\mathbb{R}$ (cf. [Bog98, Definition 2.2.1(ii)]).

We note here that trivially always $\mathcal{E}(X) \subseteq \mathcal{B}(X)$; for $\mathcal{E}(X) = \mathcal{B}(X)$, it is sufficient that X is a separable normed space (see [VTC87, Theorem I.2.2(a), p. 17]).

Associated with every Gaussian measure γ are linear mappings $a_\gamma \in (X^*)'$ and $R_\gamma : X^* \to (X^*)'$ (where $(X^*)'$ is the algebraic dual of X^*) with

$$a_\gamma(f) := \int f(x)\,\gamma(dx)$$

and

$$R_\gamma(f)(g) := \int (f(x) - a_\gamma(f))(g(x) - a_\gamma(g))\,\gamma(dx).$$

a_γ is called the **mean** and R_γ the **covariance** of γ. If $a_\gamma \equiv 0$, one calls γ **centered**.

Lemma 6.1. *Suppose that X is a separable normed space and γ is centered Gaussian on X. Then*

$$\int \|x\|^p\,\gamma(dx) < \infty$$

for every $1 \leq p < \infty$.

Proof. We learn from [Bog98, Theorem 2.8.5] that there is an $\alpha > 0$ such that

$$\int \exp(\alpha\|\cdot\|^2)\,d\gamma < \infty.$$

There exists an $N > 0$ such that $\exp(\alpha r^2) \geq r^p$ whenever $r > N$. This yields

$$\int \|\cdot\|^p\,d\gamma = \int_{[\|\cdot\|>N]} \|\cdot\|^p\,d\gamma + \int_{[\|\cdot\|\leq N]} \|\cdot\|^p\,d\gamma \leq \int \exp(\alpha\|\cdot\|^2)\,d\gamma + N^p < \infty.$$

□

Recall that a measure μ on $\mathcal{B}(X)$ is **Radon** if, for every $A \in \mathcal{B}(X)$ and every $\varepsilon > 0$, there is a compact $K_\varepsilon \subseteq A$ such that $\gamma(A \setminus K_\varepsilon) < \varepsilon$. A measure γ on $\mathcal{B}(X)$ is called **Radon Gaussian** if it is Radon and its restriction to $\mathcal{E}(X)$ is Gaussian. If X is a separable Banach space, then all Gaussian measures on X are automatically Radon Gaussian (see [VTC87, Theorem I.3.1(b), p. 29]).

For the next lemma, recall that $\mathcal{B}(X)_\gamma$ is the completion of $\mathcal{B}(X)$ with respect to γ (see Subsection 5.1.1).

Lemma 6.2. *Suppose that γ is centered Radon Gaussian on X. Let Y be another locally convex space. The distribution $\gamma \circ F^{-1}$ of any $(\mathcal{B}(X)_\gamma, \mathcal{B}(Y))$-measurable linear $F : X \to Y$ is centered Gaussian on Y.*

Lemma 6.2 is a simple consequence of [Bog98, Proposition 2.10.3] which itself has a short proof. Earlier proofs in [Vak91, KRW91] were more involved.

Proof of Lemma 6.2. Let $f \in Y^*$ be arbitrary. We have to show that $(\gamma \circ F^{-1}) \circ f^{-1}$ is Gaussian on $\mathbb{R}$. This measure is of course equal to $\gamma \circ (f \circ F)^{-1}$. Note that $f \circ F$ is a $(\mathcal{B}(X)_\gamma, \mathcal{B}(\mathbb{R}))$-measurable linear functional. [Bog98, Proposition 2.10.3] tells us that the distribution of such a functional is centered Gaussian. □

6.1.2 The structure of measures and measurable linear operators

For any Gaussian measure γ on X, denote by X^*_γ the **reproducing kernel Hilbert space** of γ, which is the closure of the set

$$\{f - a_\gamma(f) \ : \ f \in X^*\}$$

embedded into the Hilbert space $L_2(\gamma)$ (cf. [Bog98, p. 44]). Being a closed subspace of $L_2(\gamma)$, X^*_γ is itself a Hilbert space. The operator R_γ can also be defined on X^*_γ:

$$R_\gamma(f)(g) := \int f(x)(g(x) - a_\gamma(g))\,\gamma(dx), \quad f \in X^*_\gamma,\ g \in X^*.$$

If γ is a Radon Gaussian measure on X, then a_γ and $R_\gamma(f)$ ($f \in X^*_\gamma$) are evaluation functionals induced by certain points of X (see [Bog98, Theorem 3.2.3]). We shall in this case identify the functionals a_γ and $R_\gamma(f)$ with the corresponding points. This way, the mapping R_γ embeds X^*_γ into X; this embedding is continuous (see [Bog98, Corollary 3.2.4]). The embedded copy of X^*_γ in X is the **Cameron-Martin space** $H(\gamma)$ of γ. For every $h \in H(\gamma)$, we denote by $\widehat{h}$ the unique element of X^*_γ with $R_\gamma(\widehat{h}) = h$. The norm $|\cdot|_{H(\gamma)}$ on $H(\gamma)$ is given by

$$|h|_{H(\gamma)} := \left(\int |\widehat{h}|^2\,d\gamma\right)^{1/2}$$

(cf. [Bog98, p. 44 and Lemma 2.4.1]). Is it known that the Hilbert spaces X^*_γ and $H(\gamma)$ are separable (see [Bog98, Theorem 3.2.7]).

The structure of Radon Gaussian measures can be understood very well via the Cameron-Martin space. For the following result see [Bog98, Theorem 3.5.1]:

Proposition 6.3. *Let γ be a centered Radon Gaussian measure on a locally convex space X. Let $(e_n)_n$ be an orthonormal basis of $H(\gamma)$ and $(\xi_n)_n$ a sequence of independent standard Gaussian random variables over a probability space $(\Omega, \mathcal{S}, P)$. Then the series $\sum_n \xi_n e_n$ converges P-a.e. to some $(\mathcal{S}, \mathcal{B}(X))$-measurable $F : \Omega \to X$, and $\gamma = P \circ F^{-1}$.* □

We next examine the structure of measurable linear mappings. (The results will not be needed before Chapter 7.) We first quote [Bog98, Theorem 3.7.3(i), Theorem 3.7.6]:

Proposition 6.4. *Let γ be a centered Radon Gaussian measure on a locally convex space X. Let $(e_n)_n$ be an orthonormal basis of $H(\gamma)$.*

(1) Let $F : X \to X$ be a $(\mathcal{B}(X)_\gamma, \mathcal{B}(Y))$-measurable linear operator and $\mu = \gamma \circ F^{-1}$. Then F maps $H(\gamma)$ continuously onto $H(\mu)$.

(2) *Let A be a continuous linear operator on $H(\gamma)$. A extends to a $(\mathcal{B}(X)_\gamma, \mathcal{B}(X))$-measurable linear mapping $\widehat{A} : X \to X$ such that $\gamma \circ \widehat{A}^{-1}$ is Radon Gaussian and*

$$\widehat{A}x = \sum_{n=0}^{\infty} \widehat{e_n}(x) A e_n \qquad \gamma\text{-a.e.}$$

Any two $(\mathcal{B}(X)_\gamma, \mathcal{B}(X))$-measurable linear extensions of A that induce Radon measures coincide.

□

Via a simple technical trick, we can get the following result out of the preceding proposition:

Theorem 6.5. *Let γ be a centered Radon Gaussian measure on X. Let Y be another locally convex space and $F : X \to Y$ a $(\mathcal{B}(X)_\gamma, \mathcal{B}(Y))$-measurable linear mapping such that $\mu = \gamma \circ F^{-1}$ is Radon. Let $(e_n)_n$ be an arbitrary orthonormal basis of $H(\gamma)$. Then*

$$F(x) = \sum_{n=0}^{\infty} \widehat{e}_n(x) F(e_n) \qquad \gamma\text{-a.e.} \tag{6.1}$$

We believe that this fact is well-known among experts though we have not found it explicitly in the literature. The special case when X and Y are separable Banach spaces was proved in [Vak91].

In the proof, we will have to deal with the problem that the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{B}(Y)$ is in general a proper subset of $\mathcal{B}(X \times Y)$ (see [VTC87, Section I.1.3]). The following fact (which is also implicit in [VTC87, Lemma I.4.1, p. 60]) will be useful:

Lemma 6.6. *Suppose $a \in X$ and $A \in \mathcal{B}(X \times Y)$. Define the section*

$$A_a := \{y \in Y \ : \ (a, y) \in A\}.$$

Then $A_a \in \mathcal{B}(Y)$.

Proof. Fix an $a \in X$. The set

$$\{B \in \mathcal{B}(X \times Y) \ : \ B_a \in \mathcal{B}(Y)\}$$

is easily seen to form a σ-algebra. It is hence sufficient to show that it contains the open subsets of $X \times Y$, because these generate $\mathcal{B}(X \times Y)$. So let an open $B \subseteq X \times Y$ be given. The rectangular open sets form a basis of the topology of $X \times Y$, so B can be written as $\bigcup_\alpha (V_\alpha \times W_\alpha)$ for certain open sets $V_\alpha \subseteq X$ and $W_\alpha \subseteq Y$. Now $B_a = \bigcup\{W_\alpha \ : \ a \in V_\alpha\}$, so B_a is open and in particular in $\mathcal{B}(Y)$. □

Proof of Theorem 6.5. We learn from [Bog98, Example 3.1.7] that the product measure $\gamma \otimes \mu$ defined on the σ-algebra $\mathcal{B}(X) \otimes \mathcal{B}(Y)$ extends uniquely to a centered Radon Gaussian measure ρ on $\mathcal{B}(X \times Y)$; furthermore, $H(\rho)$ is the Hilbert direct sum $(H(\gamma) \times H(\mu))_2$, which means $H(\rho) = H(\gamma) \times H(\mu)$ and $|(h_1, h_2)|^2_{H(\rho)} = |h_1|^2_{H(\gamma)} + |h_2|^2_{H(\mu)}$. Let $(g_n)_n$ be an arbitrary orthonormal basis of $H(\mu)$; then $(a_n)_n$ with $a_{2n} = (e_n, 0)$ and $a_{2n+1} = (0, g_n)$ is an orthonormal basis of $H(\rho)$.

Consider the operator $G : X \times Y \to X \times Y$ with $G(x, y) := (0, F(x))$. Note that

$$(\forall\, A \in \mathcal{B}(X \times Y))\ (G^{-1}(A) = F^{-1}(A_0) \times Y) \tag{6.2}$$

where $A_0 = \{y \in Y \ :\ (0, y) \in A\}$.

We will prove three statements: (i) G is $(\mathcal{B}(X \times Y)_\rho, \mathcal{B}(X \times Y))$-measurable. (ii) The distribution $\rho \circ G^{-1}$ of G extends $\delta_0 \otimes \mu$. (iii) $\rho \circ G^{-1}$ is Radon.

Proof of (i): Let $A \in \mathcal{B}(X \times Y)$ be arbitrary. Then $G^{-1}(A) = F^{-1}(A_0) \times Y$ by (6.2). The auxiliary lemma yields $A_0 \in \mathcal{B}(Y)$, so $F^{-1}(A_0) \in \mathcal{B}(X)_\gamma$. We can write $F^{-1}(A_0)$ as the union of a $B \in \mathcal{B}(X)$ and a γ-null set N. Then

$$G^{-1}(A) = B \times Y \ \cup\ N \times Y.$$

Clearly, $B \times Y \in \mathcal{B}(X \times Y)$, and $N \times Y$ is ρ-null. This implies $G^{-1}(A) \in \mathcal{B}(X \times Y)_\rho$.

Proof of (ii): Let $A \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$ be a rectangular set, i.e. $A = V \times W$ with certain $V \in \mathcal{B}(X)$, $W \in \mathcal{B}(Y)$. Equation (6.2) yields

$$(\rho \circ G^{-1})(A) = \rho(F^{-1}(A_0) \times Y) = \gamma(F^{-1}(A_0)) = \mu(A_0) = \chi_V(0)\mu(W) = (\delta_0 \otimes \mu)(A).$$

The restriction of $\rho \circ G^{-1}$ to $\mathcal{B}(X) \times \mathcal{B}(Y)$ is hence equal to $\delta_0 \otimes \mu$ (because product measures are uniquely defined by their values on rectangles).

Proof of (iii): Let $A \in \mathcal{B}(X \times Y)$ and $\varepsilon > 0$ be arbitrary. As μ is Radon by assumption and $A_0 \in \mathcal{B}(Y)$ by the auxiliary lemma, there exists a compact $K \subseteq A_0$ such that $\mu(A_0 \setminus K) < \varepsilon$. $\{0\} \times K$ is a compact subset of A. Furthermore

$$(\rho \circ G^{-1})(A \setminus (\{0\} \times K)) = \mu(A_0 \setminus K) < \varepsilon.$$

So $\rho \circ G^{-1}$ is Radon.

Statement (i) allows us to invoke Proposition 6.4.1, which yields that G maps $H(\rho)$ continuously onto $H(\rho \circ G^{-1})$. By statements (ii) and (iii) and (again) [Bog98, Example 3.1.7], we know that $H(\rho \circ G^{-1})$ is the Hilbert direct sum of the trivial Hilbert space $\{0\}$ and $H(\mu)$; this direct sum is a subspace of $H(\rho)$, so G maps $H(\rho)$ continuously into itself. Proposition 6.4.2 yields that

$$G(x, y) = \sum_{n=0}^{\infty} \widehat{a_n}(x, y) G(a_n) \qquad \rho\text{-a.e.}$$

The left-hand side of this equation is equal to $(0, F(x))$; the right-hand side is equal ρ-a.e. to

$$\sum_{n=0}^{\infty} \widehat{e_n}(x)(0, F(e_n)) = \left(0, \sum_{n=0}^{\infty} \widehat{e_n}(x) F(e_n)\right)$$

because $G(a_n) = (0, 0)$ for odd n and $\widehat{a_n}(x, y) = \widehat{e_n}(x)$ ρ-a.e. for even n. This directly implies (6.1). $\square$

Let us introduce the following notation: If $(\Omega, \mathcal{S}, \nu)$ is a measure space, X is a normed space, $F : \Omega \to X$ is $(\mathcal{S}_\nu, \mathcal{B}(X))$-measurable, and $1 \leq p < \infty$, then

$$\|F\|_{p,\nu} := \left(\int \|F\|^p \, d\nu\right)^{1/p} .$$

The convergence in (6.1) can be strengthened in certain situations. We will deduce this from the following general result, which is found in [VTC87, Corollary 2 of Theorem V.3.2, p. 293].

Lemma 6.7. *Let X be a separable normed space and $(c_n)_n \in X^\omega$. Let $(\xi_n)_n$ be a sequence of independent random variables with mean zero on some probability space $(\Omega, \mathcal{S}, P)$. Suppose that the series $\sum_n \xi_n c_n$ converges P-a.e. to an $(\mathcal{S}, \mathcal{B}(X))$-measurable F. Then the following are equivalent for every $1 \leq p < \infty$:*

(1) $\sup_n \left\| \sum_{i=0}^n \xi_i c_i \right\|_{p,P} < \infty$*;*

(2) $\|F\|_{p,P} < \infty$*;*

(3) $\lim_{n\to\infty} \left\| F - \sum_{i=0}^n \xi_i c_i \right\|_{p,P} = 0$*.*

$\square$

Corollary 6.8. *Let γ be a centered Radon Gaussian measure on X. Let Y be a separable Banach space and $F : X \to Y$ a $(\mathcal{B}(X)_\gamma, \mathcal{B}(Y))$-measurable linear mapping. Let $(e_n)_n$ be an arbitrary orthonormal basis of $H(\gamma)$. Then the series $\sum_n \widehat{e}_n F(e_n)$ converges to F γ-a.e. and*

$$\lim_{n\to\infty} \left\| F - \sum_{i=0}^{n} \widehat{e}_i F(e_i) \right\|_{p,\gamma} = 0 \tag{6.3}$$

for every $1 \leq p < \infty$.

The special case when X is a separable Banach space, Y is a separable Hilbert space, and $p = 2$ appeared in [Wer91, Section 7.5.1].

Proof of Corollary 6.8. As Y is a separable Banach space, we have that $\gamma \circ F^{-1}$ is automatically Radon (see [VTC87, Theorem I.3.1(b), p. 29]). Theorem 6.5 hence yields that $\sum_n \widehat{e}_n F(e_n)$ converges to F γ-a.e. Consider the family $\{\widehat{e}_n\}_n$ of random variables. Any linear combination of members of this family is in X^*_γ and is hence Gaussian (see [Bog98, Lemma 2.2.8]). So $(\widehat{e}_n)_n$ is a *Gaussian process* on $\mathbb{N}$ in the sense of [Kal02, Chapter 13]. By assumption, the $\widehat{e}_n$ are pairwise orthogonal in $L_2(\gamma)$, that is, their pairwise covariance is zero. We now learn from [Kal02, Lemma 13.1] that $\{\widehat{e}_n\}_n$ is an independent family of random variables. We also know from Lemma 6.2 in connection with Lemma 6.1 that

$$\int \|F\|^p \, d\gamma = \int \|x\|^p \, (\gamma \circ F^{-1})(dx) < \infty.$$

The claim now follows from Lemma 6.7 (implication (2) $\Rightarrow$ (3)). $\square$

Remark 6.9. Under the assumptions of the previous proposition, X^*_γ is the $L_2(\gamma)$ closure of X^*. One can hence choose an orthonormal basis $(e_n)_n$ of $H(\gamma)$ such that the $(\widehat{e}_n)_n$ are elements of X^*.

6.1.3 Gaussian measures on separable Hilbert spaces

We start with a useful formula:

Lemma 6.10. *Let X be a Hilbert space, $(\xi_n)_n$ a sequence of real random variables with mean zero, variance one, and pairwise covariance zero over some probability space $(\Omega, \mathcal{S}, P)$, and $(c_n)_n \in X^\omega$ a sequence. Suppose that there is a $(\mathcal{S}, \mathcal{B}(X))$-measurable F such that*

$$\lim_{n\to\infty} \big\|F - \sum_{i=0}^{n} \xi_i c_i\big\|_{2,P} = 0.$$

Then

$$\big\|F - \sum_{i=0}^{n-1} \xi_i c_i\big\|_{2,P}^2 = \|F\|_{2,P}^2 - \sum_{i=0}^{n-1} \|c_i\|^2$$

for every $n \in \mathbb{N}$.

Proof. First note that for all $0 \le n \le m$, we have the identity

$$\big\|\sum_{i=n}^{m} \xi_i c_i\big\|_{2,P}^2 = \sum_{i,j=n}^{m} \langle c_i \mid c_j \rangle \int \xi_i \xi_j \, dP = \sum_{i=n}^{m} \|c_i\|^2.$$

Using this, we see

$$\begin{aligned}\Big\|F - \sum_{i=0}^{n-1} \xi_i c_i\Big\|_{2,P}^2 &= \lim_{m\to\infty} \Big\| \sum_{i=0}^{m} \xi_i c_i - \sum_{i=0}^{n-1} \xi_i c_i\Big\|_{2,P}^2 = \lim_{m\to\infty} \Big\| \sum_{i=n}^{m} \xi_i c_i\Big\|_{2,P}^2 \\ &= \lim_{m\to\infty} \sum_{i=n}^{m} \|c_i\|^2 = \lim_{m\to\infty} \sum_{i=0}^{m} \|c_i\|^2 - \sum_{i=0}^{n-1} \|c_i\|^2 \\ &= \lim_{m\to\infty} \Big\| \sum_{i=0}^{m} \xi_i c_i\Big\|_{2,P}^2 - \sum_{i=0}^{n-1} \|c_i\|^2 = \|F\|_{2,P}^2 - \sum_{i=0}^{n-1} \|c_i\|^2\end{aligned}$$

for every n. □

We are going to quote a result (see [Bog98, Theorem 2.3.1]) that characterizes Gaussian measures on separable Hilbert spaces and is known as the *Mourier Theorem* (sometimes also *Mourier-Prokhorov Theorem*). This characterization is via self-adjoint non-negative **nuclear** operators (see [Bog98, pp. 368-369]). A self-adjoint operator $A : X \to X$ on a separable Hilbert space X is nuclear if, and only if, the series

$$\sum_n \langle Ae_n \mid e_n \rangle$$

converges for every orthonormal basis $(e_n)_n$ of X; in this case the sum of the series does not depend on the choice of the orthonormal basis and is called the **trace** of A.

Suppose that X is a separable Hilbert space and γ is Gaussian on X. We will identify X^* and X; so we can define R_γ on X. Also note that γ is necessarily Radon Gaussian (see above), so for every $x \in X$, the function $R_\gamma(x)$ can be identified with a point in X.

Proposition 6.11 (Mourier Theorem)**.** *The Gaussian measures γ on a separable Hilbert space X correspond one-to-one to the pairs (a, K), where $a \in X$ and $K : X \to X$ is self-adjoint non-negative nuclear, via the correspondence*

$$a_\gamma = a \quad \textit{and} \quad R_\gamma = K.$$

□

In the proof of [Bog98, Theorem 2.3.1], the following identity is shown, which we will need below:

Lemma 6.12. *Let γ be a centered Gaussian measure on a separable Hilbert space X. Then*

$$\mathrm{trace}(R_\gamma) = \int \|x\|^2 \, \gamma(dx).$$

□

6.2 An effective version of Mourier's Theorem

Our aim is to prove an effective version of Proposition 6.11. We can simplify things by only considering measures on the canonical real computable Hilbert space $(\ell_2, \|\cdot\|, e)$ (see Subsection 2.4.4).

Denote by $\mathcal{G}$ the class of all Gaussian measures on ℓ_2; denote by $\mathcal{G}_c$ the subclass of all centered measures. Let Γ_{top} (the "topological representation") be the restriction of $\theta^e_{\mathcal{M}_0<}$ to $\mathcal{G}$. Another representation Γ_{alg} (the "algebraic representation") is suggested by Proposition 6.11:

$$\Gamma_{\text{alg}}\langle r, s, t\rangle_{\omega\omega\omega} = \gamma \quad :\Longleftrightarrow \quad \begin{bmatrix} \alpha^e_{\text{Cauchy}}(r) = a_\gamma, \\ \delta^{e,e}_{\text{ev}}(s) = R_\gamma, \\ \rho_{\mathbb{R}>}(t) = \operatorname{trace}(R_\gamma). \end{bmatrix}$$

We introduce another representation Γ_{weak} of $\mathcal{G}_c$ with $\Gamma_{\text{alg}}|^{\mathcal{G}_c} \le \Gamma_{\text{weak}}$:

$$\Gamma_{\text{weak}}(r) = \gamma \quad :\Longleftrightarrow \quad [\alpha^e_{\text{Cauchy}} \to \rho_{\mathbb{R}}]_{\text{TOT}} = (x \mapsto \langle R_\gamma x \mid x\rangle).$$

Note that a Γ_{weak}-name of a measure $\gamma \in \mathcal{G}_c$ provides enough information to compute the form $(x, y) \mapsto \langle R_\gamma x \mid y\rangle$ via the well-known formula

$$\langle R_\gamma x \mid y\rangle = \frac{1}{4}\big(\langle R_\gamma(x+y) \mid x+y\rangle + \langle R_\gamma(x-y) \mid x-y\rangle\big).$$

Γ_{weak} will not play an important role in this chapter, but in Chapter 7. However, we will use Γ_{weak} in the formulation of Lemma 6.15 below, because it will be useful once more in that chapter.

Theorem 6.13 (Effective Mourier Characterization)**.** $\Gamma_{\text{top}} \equiv \Gamma_{\text{alg}}$.

The proof splits into a number of lemmas:

Let $\mathbb{R}^\omega$ be the ω-fold product of $\mathbb{R}$ equipped with the product topology. Let $(\pi_n)_n$ be the sequence of natural projections from $\mathbb{R}^\omega$ to $\mathbb{R}$. If ϑ is a canonical numbering of the set

$$\beta_\vartheta := \{\pi^{-1}_{n_1}((r_1; s_1)) \cap \cdots \cap \pi^{-1}_{n_m}((r_m; s_m)) \\ \qquad : \; m \ge 1, \; (\forall\, 1 \le i \le m)\,(n_i \in \mathbb{N}, \, r_i, s_i \in \mathbb{Q}, \, r_i < s_i)\}, \quad (6.4)$$

then $(\mathbb{R}^\omega, \vartheta)$ is a computable T_0-space. It is easy to check that $(\pi_n)_n$ is $[\vartheta_{\text{std}} \to \rho_{\mathbb{R}}]^\omega$-computable.

Let γ_0 be the standard Gaussian measure on $\mathbb{R}$. Consider the product measure γ_0^ω on the σ-algebra $\mathcal{B}(\mathbb{R})^\omega$. This σ-algebra is equal to $\mathcal{B}(\mathbb{R}^\omega)$ by [VTC87, Proposition I.1.7(b)], so γ_0^ω is a Borel measure on $\mathbb{R}^\omega$.

Lemma 6.14. *γ_0^ω is $\vartheta_{\mathcal{M}_0}$-computable.*

Proof. Consider the definition of the $\vartheta_{\mathcal{M}_0}$-representation in Subsection 2.5.2. We have to compute a sequence $(U_n)_n$ as in (2.11). We shall show that we can simply take ϑ as this sequence: To this end, we only need to demonstrate how to compute the γ_0^ω-contents of algebraic expression of sets from β_ϑ. These sets are finite intersections of sets of the form

$$\pi_n^{-1}((r;s)), \qquad r, s \in \mathbb{Q},\, r < s, \tag{6.5}$$

so it is sufficient to demonstrate how to compute the γ_0^ω-content of algebraic expressions of sets of the form (6.5). Let A be a set whose description as such an algebraic expression is given. By first transforming the algebraic expression into disjunctive normal form and then using the principle of inclusion and exclusion, we can effectively reduce the computation of $\gamma_0^\omega(A)$ to the computation of the γ_0^ω-content of sets of the form

$$A_1 \cap \cdots \cap A_m \cap (X \setminus B_1) \cap \cdots \cap (X \setminus B_k),$$

where each A_i and each B_j has the form (6.5). By sorting the A_i, B_j by the respective n from (6.5), we can compute algebraic expressions $C_1, \ldots, C_\ell$ of open intervals with rational endpoints such that

$$A = \pi_{n_1}^{-1}(C_1) \cap \cdots \cap \pi_{n_\ell}^{-1}(C_\ell).$$

Then

$$\gamma_0^\omega(A) = \gamma_0(C_1) \cdot \cdots \cdot \gamma_0(C_\ell).$$

The computation of the numbers $\gamma_0(C_k)$ is easily seen to be reducible to the computation of the integral of the standard Gaussian density over given intervals with rational endpoints. This integration can be computed by standard results in computable analysis (see [Wei00]). $\square$

Lemma 6.15. *The total multi-valued mapping* $\mathrm{ONB} : \mathcal{G}_c \rightrightarrows (\ell_2)^\omega$ *with*

$$\mathrm{ONB}(\gamma) \ni (b_n)_n \quad :\Longleftrightarrow \quad \textit{the sequence } (\langle b_n \mid \cdot \rangle)_n \textit{ is an orthonormal basis of } (\ell_2)^*_\gamma$$

is total and $(\Gamma_{\mathrm{weak}}, [\alpha^e_{\mathrm{Cauchy}}]^\omega)$*-computable.*

Proof. Let a Γ_{weak}-name of a measure $\gamma \in \mathcal{G}_c$ be given. For any two points $x, y \in \ell_2$, we have

$$\|\langle x \mid \cdot \rangle - \langle y \mid \cdot \rangle\|^2_{2,\gamma} = \int \langle x - y \mid \cdot \rangle^2 \, d\gamma \leq \|x - y\|^2 \int \|\cdot\|^2 \, d\gamma = \|x - y\|^2 \mathrm{trace}(R_\gamma)$$

So the completeness of e in ℓ_2 implies the completeness of $(\langle e(n) \mid \cdot \rangle)_n)$ in $(\ell_2)^*_\gamma$. By means of our input information on γ, we can $[\alpha^e_{\mathrm{Cauchy}}, \rho_\mathbb{R}]$-compute the mapping

$$x \mapsto \sqrt{\langle R_\gamma x \mid x \rangle},$$

i.e. we can compute the norm of $\langle x \mid \cdot \rangle$ in $(\ell_2)^*_\gamma$. So

$$((\ell_2)^*_\gamma, \|\cdot\|_{2,\gamma}, (\langle e(n) \mid \cdot \rangle)_n)$$

is a computable Banach space (relative to our input information). We apply the Effective Independence Lemma[1] to compute a sequence $(n_i)_i$ such that the sequence $(\langle e(n_i) \mid \cdot \rangle)_i$ is independent and complete in $(\ell_2)^*_\gamma$. By Gram-Schmidt orthonormalization with respect to $\langle R_\gamma \cdot \mid \cdot \rangle$, we can then compute a sequence $(b_n)_n \in \mathrm{ONB}(\gamma)$. $\square$

Lemma 6.16. *The total multi-valued mapping $S : \mathcal{G} \rightrightarrows \Lambda(\vartheta_{\mathrm{std}} \to \alpha^e)^{\gamma_0^\omega}_{\mathrm{MEAN}}$ with*

$$S(\gamma) \ni [F]_{\gamma_0^\omega} \quad :\Longleftrightarrow \quad \gamma = \gamma_0^\omega \circ F^{-1}$$

is well-defined and $(\Gamma_{\mathrm{alg}}, [\vartheta_{\mathrm{std}} \to \alpha^e]^{\gamma_0^\omega}_{\mathrm{MEAN}})$-computable.

Proof. Let μ be the centered Gaussian measure on ℓ_2 with $R_\mu = R_\gamma$. We invoke Lemma 6.15 to compute a sequence $(b_n)_n$ such that $(\langle b_n \mid \cdot \rangle)_n$ is an orthonormal basis of X^*_μ. Then $(R_\mu b_n)_n$ is an orthonormal basis of $H(\mu)$. Proposition 6.3 tells us that there is a $G : \mathbb{R}^\omega \to \ell_2$ such that $\mu = \gamma_0^\omega \circ G^{-1}$ and

$$G = \sum_{n=0}^{\infty} \pi_n R_\gamma b_n \qquad \gamma_0^\omega\text{-a.e.}$$

So $\gamma = \gamma_0^\omega \circ F^{-1}$, where $F := a_\gamma + G$. It is sufficient to demonstrate how to compute a $(\vartheta_{\mathrm{std}}, \alpha^e)^{\gamma_0^\omega}_{\mathrm{MEAN}}$-realization of F. Such a realization of F, however, can easily be computed from a $(\vartheta_{\mathrm{std}}, \alpha^e)^{\gamma_0^\omega}_{\mathrm{MEAN}}$-realization of G. It remains to compute such a realization of G.

Put

$$G_n := \sum_{i=0}^{n-1} \pi_i R_\gamma b_i.$$

Lemma 6.1 yields

$$\|G\|^2_{2,\gamma_0^\omega} = \int \|x\|^2 \, \mu(dx) < \infty,$$

so we get from Lemma 6.7 (implication (2) $\Rightarrow$ (3)):

$$\lim_{n\to\infty} \|G - G_n\|_{2,\gamma_0^\omega} = 0.$$

Lemma 6.10 then yields

$$\|G - G_n\|^2_{2,\gamma_0^\omega} = \sum_{i=n}^{\infty} \|R_\gamma b_i\|^2$$

for every $n \in \mathbb{N}$. This equality (for $n = 0$) and Lemma 6.12 yield

$$\mathrm{trace}(R_\gamma) = \sum_{i=0}^{\infty} \|R_\gamma b_i\|^2.$$

[1]To be technically sound here, we should be working with representations of Banach spaces, similar as recently in [GM08], and accordingly reformulate the results of Section 2.4 in a more uniform way. This, however, would cause a large technical overhead. The proofs in Section 2.4 yield that all algorithms are sufficiently uniform.

As we are given a $\delta_{\mathrm{ev}}^{e,e}$-name of R_γ and a $\rho_{\mathbb{R}>}$-name of $\mathrm{trace}(R_\gamma)$, we can compute a function $m : \mathbb{N} \to \mathbb{N}$ such that

$$\left\|G - G_{m(n)}\right\|_{2,\gamma_0^\omega}^2 = \sum_{i=m(n)}^{\infty} \|R_\gamma b_i\|^2 < 2^{-n}.$$

We combine this with Lyapunov's inequality (see e.g. [Shi96]) and get

$$\left\|G - G_{m(n)}\right\|_{1,\gamma_0^\omega} \leq \left\|G - G_{m(n)}\right\|_{2,\gamma_0^\omega} < 2^{-n/2}.$$

As we can $[\vartheta_{\mathrm{std}} \to \alpha_{\mathrm{Cauchy}}^e]^\omega$-compute the sequence $(G_k)_k$, we can use Proposition 5.21.1 and compute a sequence of functions $\phi^{(k)} :\subseteq \mathbb{N} \times \mathbb{N}^\omega \to \mathbb{N}$ such that $\phi^{(k)}$ is a $[\vartheta_{\mathrm{std}} \to \alpha^e]_{\mathrm{MEAN}}^{\gamma_0^\omega}$-realization of G_k for every k. Put $\Phi(n,p) := \phi^{(m(2n+2))}(n+1,p)$ for every $n \in \mathbb{N}$, $p \in \mathrm{dom}(\vartheta_{\mathrm{std}})$. Then

$$\begin{aligned}
&\int^* e(G, \vartheta_{\mathrm{std}}, \Phi_n, x)\, \gamma_0^\omega(dx) \\
&\leq \|G - G_{m(2n+2)}\|_{1,\gamma_0^\omega} + \int^* e(G_{m(2n+2)}, \vartheta_{\mathrm{std}}, \phi_{n+1}^{(m(2n+2))}, x)\, \gamma_0^\omega(dx) \\
&\leq 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}
\end{aligned}$$

for every n, so Φ is a $[\vartheta_{\mathrm{std}} \to \alpha^e]_{\mathrm{MEAN}}^{\gamma_0^\omega}$-realization of G. □

We are ready to prove the first half of Theorem 6.13:

Proof of Theorem 6.13 (part 1 of 2). We prove $\Gamma_{\mathrm{alg}} \leq \Gamma_{\mathrm{top}}$. Suppose we are given a Γ_{alg}-name of a measure $\gamma \in \mathcal{G}$. By the previous lemma, we can $[\vartheta_{\mathrm{std}} \to \alpha^e]_{\mathrm{MEAN}}^{\gamma_0^\omega}$-compute an $[F]_{\gamma_0^\omega}$ such that $\gamma = \gamma_0^\omega \circ F^{-1}$. In view of Proposition 5.19.3, we can also $[\vartheta_{\mathrm{std}} \to \alpha^e]_{\mathrm{APP}}^{\gamma_0^\omega}$-compute $[F]_{\gamma_0^\omega}$. γ_0^ω is $\vartheta_{\mathcal{M}_0}$-computable by Lemma 6.14, so we can invoke Theorem 5.32 to compute a Γ_{top}-name of γ. □

We present two estimates taken from [TWW88, Lemma A.2.9.1, Lemma A.2.9.2]:

Lemma 6.17. *For every $\gamma \in \mathcal{G}$, one has*

$$\gamma(B(a_\gamma, 1)) \leq \frac{4}{3}\psi\left(\frac{2}{\sqrt{\mathrm{trace}(R_\gamma)}}\right),$$

where $\psi(x) := \sqrt{2/\pi}\int_0^x \exp(-t^2/2)\,dt$. □

Lemma 6.18. *For every $\gamma \in \mathcal{G}$ and $r > 0$ one has*

$$\gamma(\ell^2 \setminus B(a_\gamma, r)) \leq 5\exp\left(-\frac{r^2}{2\,\mathrm{trace}(R_\gamma)}\right).$$

□

Lemma 6.19. *The mapping $\gamma \mapsto a_\gamma$ for $\gamma \in \mathcal{G}$ is $(\Gamma_{\mathrm{top}}, \alpha^e_{\mathrm{Cauchy}})$-computable.*

Proof. Let a Γ_{top}-name of some measure $\gamma \in \mathcal{G}$ be given. It is sufficient to demonstrate how to $\alpha^e_{\mathrm{Cauchy}}$-compute a 2^{-k}-approximation to a_γ uniformly in k.

Our first task is to find upper bounds on $\|a_\gamma\|$ and $\mathrm{trace}(R_\gamma)$. Note that the Γ_{top}-name of γ contains sufficient information to $(\alpha^e, \rho_{\mathbb{R}<})$-compute the map $a \mapsto \gamma(B(a,1))$ for $a \in \mathcal{R}_{\alpha^e}$; so we can effectively search for an $a_0 \in \mathcal{R}_{\alpha^e}$ and a number $c \in \mathbb{Q} \cap (0;1)$ such that $\gamma(B(a_0,1)) \geq c$. We then know from Anderson's inequality (see [Bog98, Theorem 2.8.10]) that $\gamma(B(a_\gamma,1)) \geq c$. We effectively search for an $R \in \mathbb{N}$ such that $\gamma(B(0,R-1)) > 1-c$. Then $\gamma(B(a_0,1))$ and $\gamma(B(0,R-1))$ must have nonempty intersection, which implies

$$\|a_\gamma\| < R.$$

Lemma 6.17 yields that

$$c \leq \frac{4}{3}\psi\left(\frac{2}{\sqrt{\mathrm{trace}(R_\gamma)}}\right),$$

where $\psi : [0,\infty) \to \mathbb{R}$ is a computable strictly increasing function. We can hence compute $s := \psi^{-1}(3c/4)$ (cf. [Wei00, Exercise 6.3.7]) and have

$$\mathrm{trace}(R_\gamma) \leq 4/s^2 =: M.$$

Via Lemma 6.18, we get the following estimate for every $m \in \mathbb{N}$, $m > R$:

$$\begin{aligned}
&\int_{\ell_2 \setminus B(0,m)} \|x\|\,\gamma(dx) \leq \int_{\ell_2 \setminus B(a_\gamma, m-R)} \|x\|\,\gamma(dx) \\
&= \sum_{i=m}^{\infty} \int_{B(a_\gamma, i+1-R)\setminus B(a_\gamma, i-R)} \|x\|\,\gamma(dx) \leq \sum_{i=m}^{\infty} (i+1)\gamma(\ell_2 \setminus B(a_\gamma, i-R)) \\
&\leq 5\sum_{i=m}^{\infty}(i+1)\exp\left(-\frac{(i-R)^2}{2\,\mathrm{trace}(R_\gamma)}\right) \leq 5\sum_{i=m}^{\infty}(i+1)\exp\left(-\frac{(i-R)^2}{2M}\right).
\end{aligned}$$

Using standard methods from analysis, one can find a computable upper bound $\phi_{M,R}(m)$ for the latter term that converges to zero in m. We can hence effectively find an m_0 such that

$$\int_{\ell_2 \setminus B(0,m_0)} \|x\|\,\gamma(dx) < 2^{-k}.$$

It follows directly from the definitions that a_γ is equal to the Pettis integral $\int x\,\gamma(dx)$. Choose a $(\rho_{\mathbb{R}}, \rho_{\mathbb{R}})$-computable function $h_{m_0} : \mathbb{R} \to \mathbb{R}$ with

$$\chi_{(-\infty;m_0)} \leq h_{m_0} \leq \chi_{(-\infty;m_0+1)}.$$

Put $g_{m_0}(x) := xh_{m_0}(\|x\|)$ for every $x \in \ell_2$. We have

$$\begin{aligned}
&\|a_\gamma - \int g_{m_0}\, d\gamma\| = \|\int x\,\gamma(dx) - \int xh_{m_0}(\|x\|)\,\gamma(dx)\| \\
&\le \int (1 - h_{m_0}(\|x\|))\|x\|\,\gamma(dx) \le \int (1 - \chi_{(-\infty;m_0)}(\|x\|))\|x\|\,\gamma(dx) \\
&= \int_{\ell_2 \setminus B(0,m_0)} \|x\|\,\gamma(dx) < 2^{-k}.
\end{aligned}$$

It remains to demonstrate how to α^e_{Cauchy}-compute $\int g_{m_0}\, d\gamma$. We can clearly $[\theta^e_{\text{std}} \to \alpha^e_{\text{Cauchy}}]$-compute g_{m_0}, so we can also $[\theta^e_{\text{std}} \to \alpha^e]^\gamma_{\text{MEAN}}$-compute $[g_{m_0}]_\gamma$. The norm of g_{m_0} is bounded by m_0+1, so we have all necessary input to invoke Theorem 5.27.2 (in the form of Corollary 5.28) and compute $\int g_{m_0}\, d\gamma$. $\square$

Lemma 6.20. *The mapping $\gamma \mapsto \operatorname{trace}(R_\gamma)$ for $\gamma \in \mathcal{G}$ is $(\Gamma_{\text{top}}, \rho_{\mathbb{R}})$-computable.*

Proof. Let a Γ_{top}-name of some measure $\gamma \in \mathcal{G}$ be given. It is sufficient to demonstrate how to $\rho_{\mathbb{R}}$-compute a 2^{-k}-approximation to a_γ uniformly in k.

We can compute a_γ by the previous lemma. We can then $[\theta^e_{\text{std}} \to \theta^e_{\text{std}}]$-compute the mapping $x \mapsto x - a_\gamma$. In combination with Lemma 2.11.1, this yields that we can $[\theta^e_{\mathcal{O}<} \to \theta^e_{\mathcal{O}<}]$-compute $U \mapsto U + a_\gamma$. We can hence Γ_{top}-compute the measure γ' with $\gamma'(A) := \gamma(A + a_\gamma)$. Note that $R_{\gamma'} = R_\gamma$, but $a_{\gamma'} = 0$. Thus we can assume in the following that $a_\gamma = 0$.

As in the proof of the previous lemma, we can effectively find an upper bound M for $\operatorname{trace}(R_\gamma)$. Via Lemma 6.18, we get the following estimate for every $m \in \mathbb{N}$, $m \ge 1$:

$$\begin{aligned}
&\int_{\ell^2 \setminus B(0,m)} \|x\|^2\,\gamma(dx) = \sum_{i=m}^{\infty} \int_{B(0,i+1)\setminus B(0,i)} \|x\|^2\,\gamma(dx) \\
&\le \sum_{i=m}^{\infty} (i+1)^2 \gamma(\ell^2 \setminus B(0,i)) \le 5 \sum_{i=m}^{\infty} (i+1)^2 \exp\left(-\frac{i^2}{2\operatorname{trace}(R_\gamma)}\right) \\
&\le 5 \sum_{i=m}^{\infty} (i+1)^2 \exp(-i^2/(2M)).
\end{aligned}$$

Using standard methods from analysis, one can find a computable upper bound $\phi_M(m)$ for the latter term that converges to zero in m. We can hence effectively find an m_0 such that

$$\int_{\ell_2 \setminus B(0,m_0)} \|x\|^2\,\gamma(dx) < 2^{-k}.$$

It follows from Lemma 6.12 that $\operatorname{trace}(R_\gamma) = \int \|x\|^2\,\gamma(dx)$. Choose a $(\rho_{\mathbb{R}}, \rho_{\mathbb{R}})$-computable function $h_{m_0} : \mathbb{R} \to \mathbb{R}$ with

$$\chi_{(-\infty;m_0)} \le h_{m_0} \le \chi_{(-\infty;m_0+1)}.$$

Put $g_{m_0}(x) := \|x\|^2 h_{m_0}(\|x\|)$ for every $x \in \ell_2$. We have

$$\begin{aligned}
&\big|\mathrm{trace}(R_\gamma) - \int g_{m_0}\, d\gamma\big| = \big| \int \|x\|^2\, \gamma(dx) - \int \|x\|^2 h_{m_0}(\|x\|)\, \gamma(dx)\big| \\
&= \int (1 - h_{m_0}(\|x\|))\|x\|^2\, \gamma(dx) \leq \int (1 - \chi_{(-\infty;m_0)}(\|x\|))\|x\|^2\, \gamma(dx) \\
&= \int_{\ell_2 \setminus B(0,m_0)} \|x\|^2\, \gamma(dx) < 2^{-k}.
\end{aligned}$$

It remains to demonstrate how to α^e_{Cauchy}-compute $\int g_{m_0}\, d\gamma$. This object can be computed analogously to the object with the same name in the previous lemma's proof. □

Lemma 6.21. *The mapping $\gamma \mapsto R_\gamma$ for $\gamma \in \mathcal{G}$ is $(\Gamma_{\text{top}}, \delta^{e,e}_{\text{ev}})$-computable.*

Proof. By type conversion, it is sufficient to show that $(\gamma, y) \mapsto R_\gamma y$ is $([\Gamma_{\text{top}}, \alpha^e_{\text{Cauchy}}], \alpha^e_{\text{Cauchy}})$-computable. Let a Γ_{top}-name of some measure $\gamma \in \mathcal{G}$ and an α^e_{Cauchy}-name of some $y \in \ell_2$ be given. It is sufficient to demonstrate how to $\rho_{\mathbb{R}}$-compute a 2^{-k}-approximation to $R_\gamma y$ uniformly in k.

As in the previous lemma's proof, we can reduce the computation to the case that $a_\gamma = 0$. By the previous lemma's assertion, we can compute an upper bound M of $\mathrm{trace}(R_\gamma)$. Let ϕ_M be the function with the same name from the previous lemma's proof. We get for every $m \in \mathbb{N}$, $m \geq 1$:

$$\int_{\ell_2 \setminus B(0,m)} \|\langle x \mid y \rangle x\|\, \gamma(dx) \leq \|y\| \int_{\ell_2 \setminus B(0,m)} \|x\|^2\, \gamma(dx) \leq \|y\| \phi_M(m).$$

We can hence effectively find an m_0 such that

$$\int_{\ell_2 \setminus B(0,m_0)} \|\langle x \mid y \rangle x\|\, \gamma(dx) < 2^{-k}.$$

It follows directly from the definitions that $R_\gamma y$ is equal to the Pettis integral

$$\int \langle y \mid x \rangle x\, \gamma(dx).$$

The rest of the proof is analogous to the last parts of the previous two lemmas' proofs. □

Proof of Theorem 6.13 (part 2 of 2). Combining the previous three lemmas yields $\Gamma_{\text{top}} \leq \Gamma_{\text{alg}}$. □

Chapter 7

Application: Are Unbounded Linear Operators Computable on the Average for Gaussian Measures?

7.1 A question of Traub and Werschulz

For convenience, let us explicitly state the following consequence of Corollary 6.8 and Remark 6.9:

Proposition 7.1. *Suppose that X and Y are real separable Banach spaces, γ is a centered Gaussian measure on X, $F : X \to Y$ is linear and $(\mathcal{B}(X)_\gamma, \mathcal{B}(Y))$-measurable. For every $\varepsilon > 0$, there exist $n \in \mathbb{N}$, $a_1, \dots, a_n \in Y$, and $f_1, \dots, f_n \in X^*$ such that the mapping $\Phi : X \to Y$ with*

$$\Phi(x) := \sum_{i=1}^{n} a_i f_i(x), \qquad x \in X, \tag{7.1}$$

fulfills

$$\int \|F - \Phi\|^2 \, d\gamma < \varepsilon.$$

□

Werschulz [Wer87] proved this result for the case that X and Y are separable Hilbert spaces and additionally assuming

$$\int \|F\|^2 \, d\gamma < \infty.$$

It was later found independently in [KRW91] and [Vak91] that this additional assumption is automatically fulfilled (because the distribution of F is necessarily Gaussian). In [Wer91], the requirement that X is separable Hilbert is weakened to the requirement the X is separable Banach.

The history of these results was reviewed in the survey article [TW94] and in a chapter of the book [TW98]. Why did the results draw so much attention? We have already discussed in Chapters 1 and 4 the practical importance and inherent difficulty of computing linear unbounded operators. Werschulz [Wer87] proved that no real-number machine[1] with access to a finite number of oracles for continuous linear functionals can approximate such an operator with finite worst-case error. It is, however, very easy to implement mappings of the form (7.1) on real-number machines of the described type. So if the unbounded linear operator to be approximated is measurable (which is a very weak requirement) and its input can be assumed to be Gaussian distributed, the Proposition 7.1 yields that real-number machines of the described type can approximate the operator with arbitrarily small average quadratic error. (Note, however, that these machines are neither uniform in the operator nor in the error level. We will get back to this point below.) In the language of information based-complexity (IBC) (see [TWW88]) this reads: *Linear approximation problems are solvable on the average for Gaussian measures.*

Traub and Werschulz [TW98] compare Werschulz' negative result for the worst-case setting to Pour-El and Richards' [PER89] First Main Theorem, which says that unbounded linear operators typically map some computable points to uncomputable points (cf. Lemma 2.30 and Theorem 2.31). As the transition from the worst-case setting to the Gaussian average-case setting makes the approximation of unbounded linear operators a *solvable problem* in the sense of IBC, Traub and Werschulz ask whether such a transition is also possible in Turing machine-based computability. More precisely, they pose the following question [TW98, p. 60]:

> Is every (measurable) linear operator computable on the average for Gaussian measures?

This chapter is devoted to the discussion of this question.

Before we turn to formal considerations, we would like to comment on a certain philosophical issue and its connection to Traub and Werschulz' question: The First Main Theorem can be used to construct computable initial conditions for the three-dimensional wave equation such that the unique solution at time one is uncomputable [PER89, PEZ97]. This example spawned questions on the potential computational power of physical devices (see e.g. [Pen89]) and hence on the validity of the Church-Turing thesis. The question is: Can one build a wave computer that computes more functions than the Turing machine? In an online article[2] from 2001 the mathematical physicist Freeman Dyson said:

> Marian Pour-El and Ian Richards, two mathematicians at the University of Minnesota, proved a theorem twenty years ago that says, in a mathematically precise

[1] See [Nov95] for an appropriate formal machine model.

[2] http://www.edge.org/3rd_culture/dyson_ad/dyson_ad_index.html

> way, that analog computers are more powerful than digital computers. They give examples of numbers that are proved to be non-computable with digital computers but are computable with a simple kind of analog computer.

Joseph Traub posted the following reply[3]:

> The Pour-El and Richards result is for a worst case setting. It's been shown that there's no difficulty "on the average". [...] The bad result is just an artifact of insisting on certainty.

Regarding this point, we would like to give our opinion that the average-case setting is of little relevance in this philosophical discussion because a potential wave computer would depend on the possibility to configure initial conditions for the wave equation with infinite precision; in the average-case setting, however, one models a situation in which initial conditions come in randomly from some source. Anyway, there are other arguments that it is very unlikely that a wave hyper-computer can be built (see [ZW03]). One can furthermore show that wave propagation and a number of other physical processes *are* computable with respect to certain physically reasonable representations (see e.g. [ZW03, WZ05, WZ06b]).

7.2 Possible answers

In the language of IBC, one has the result that linear approximation problems are solvable on the average because for every linear measurable operator $F : X \to Y$ of separable Banach spaces, every centered Gaussian measure γ on X, and every error level ε, there exists a real-number machine that computes a mapping whose $\|\cdot\|_{2,\gamma}$-distance to F is smaller than ε. So the machine may depend on the operator, the measure, and the error level. If one settles for this level of non-uniformness in computable analysis, too, one has the following corollary to Proposition 7.1:

Proposition 7.2. *Suppose that $(X, \|\cdot\|, e)$ and $(Y, \|\cdot\|, h)$ are computable Banach spaces, γ is a centered Gaussian measure on X, $F : X \to Y$ is linear and $(\mathcal{B}(X)_\gamma, \mathcal{B}(Y))$-measurable. For every $\varepsilon > 0$, there exists an $n \in \mathbb{N}$, α^h_{Cauchy}-computable $a_1, \ldots, a_n \in Y$, $[\alpha^e_{\text{Cauchy}} \to \rho_{\mathbb{R}}]$-computable $f_1, \ldots, f_n \in X^*$ such that*

$$\Big\|F - \sum_{i=1}^{n} a_i f_i\Big\|_{2,\gamma} < \varepsilon.$$

[3] http://www.edge.org/discourse/analog_digital.html#traub

Proof. In view of Proposition 7.1, it is sufficient to show that for any

$$\widetilde{a}_1, \ldots, \widetilde{a}_n \in Y, \qquad \widetilde{f}_1, \ldots, \widetilde{f}_n \in X^*, \quad \text{and} \quad \varepsilon > 0,$$

there are α^h_{Cauchy}-computable $a_1, \ldots, a_n \in Y$ and $[\alpha^e_{\text{Cauchy}} \to \rho_{\mathbb{R}}]$-computable $f_1, \ldots, f_n \in X^*$ such that

$$\Big\| \sum_{i=1}^n \widetilde{a}_i \widetilde{f}_i - \sum_{i=1}^n a_i f_i \Big\|_{2,\gamma} < \varepsilon.$$

As

$$\Big\| \sum_{i=1}^n \widetilde{a}_i \widetilde{f}_i - \sum_{i=1}^n a_i f_i \Big\|_{2,\gamma} \leq \sum_{i=1}^n \|\widetilde{a}_i \widetilde{f}_i - a_i f_i\|_{2,\gamma},$$

this reduces to the case $n = 1$. As

$$\begin{aligned} \|\widetilde{a}\widetilde{f} - af\|_{2,\gamma} &\leq \|\widetilde{a}\widetilde{f} - \widetilde{a}f\|_{2,\gamma} + \|\widetilde{a}f - af\|_{2,\gamma} \\ &= \|\widetilde{a}\| \|\widetilde{f} - f\|_{2,\gamma} + \|\widetilde{a} - a\| \|f\|_{2,\gamma}, \end{aligned}$$

it is sufficient to show that the α^h_{Cauchy}-computable points are dense in Y and the $[\alpha^e_{\text{Cauchy}} \to \rho_{\mathbb{R}}]$-computable linear functionals are dense in X^*_γ. The former is obvious. For the latter, we learn from [Bog98, Proposition 3.1.9] that we only need to show that the computable functionals separate the points in X, i.e. that for every $x \in X \setminus \{0\}$ there is a computable $f \in X^*$ with $f(x) \neq 0$. Put, for abbreviation, $x_n := \alpha^e(n)$ for every n. Then $\{x_n : n \in \mathbb{N}\}$ is dense in X. For every n, let $f_n :\subseteq X \to \mathbb{R}$ be the unique linear functional with $\operatorname{dom}(f_n) = \operatorname{span}_{\mathbb{R}}\{x_n\}$ and $f_n(x_n) = \|x_n\|$. Note that $\|f_n\| = 1$. Each f_n is easily seen to be computable. By the Effective Hahn-Banach Theorem of [MN82], we can extend each f_n to a computable linear functional on X (which we will also denote by f_n) such that $\|f_n\| \leq 2$. We show that the computable functionals f_n obtained in this way separate the points in X: Let $x \in X \setminus \{0\}$ be arbitrary. There exists an n such that $0 < \|x_n\| - 2\|x_n - x\|$. We have the estimate

$$\|x_n\| = |f_n(x_n)| \leq |f_n(x)| + |f_n(x_n - x)| \leq |f_n(x)| + 2\|x_n - x\|$$

and thus

$$0 < \|x_n\| - 2\|x_n - x\| \leq |f_n(x)|.$$

□

It is, however, natural in computable analysis to seek for uniform algorithms. Our results in this direction will be of a negative nature, even though we will restrict ourselves to unbounded linear operators on separable Hilbert spaces that are inverses of computable operators. We have already seen in Chapter 1 that this is a common type of problem in applications. We will hence reconsider the generalized inversion problem from Chapter 4, but this time only seeking for algorithms that work well on the average. Loosely speaking, the question is: Given a linear operator on ℓ_2, its adjoint, and a centered Gaussian measure on ℓ_2, can we uniformly compute the operator's generalized inverse up to an arbitrarily small average error? We have not yet said what we understand by *computable on the average*; we will consider several interpretations of this notion. Our results can be sketched as follows:

(I) Even if we interpret "computable on the average" in a rather weak sense and restrict ourselves to a single very simple Gaussian measure and to injective self-adjoint operators T with dense range, the operation $T \mapsto T^\dagger$ $(= T^{-1})$ is Σ_2^0-hard.

(II) Even if we interpret "computable on the average" in a rather strong sense, the operation $T \mapsto T^\dagger$ is Σ_2^0-computable uniformly in T, T^*, and the Gaussian measure.

These results show that uncomputability is still present in the average-case setting. The degree of uncomputability is actually just the same as in the worst-case setting studied in Chapter 4. Our proofs will rely on results from Chapters 4, 5, and 6. Traub and Werschulz' question was in fact the original impetus for studying the subjects treated in those chapters.

7.2.1 Generalized inverses as Gaussian random elements

The technical results corresponding to the statements (I) and (II) above will be proved in Subsections 7.2.2 and 7.2.3. In preparation, we shall make a few observations on the structure of generalized inverses considered as Gaussian random elements.

Lemma 7.3. *Suppose* $T \in B(\ell_2)$. *Then* $\mathrm{dom}(T^\dagger) \in \mathcal{B}(\ell_2)$ *and* $T^\dagger$ *is* $(\mathrm{dom}(T^\dagger) \cap \mathcal{B}(\ell_2), \mathcal{B}(\ell_2))$-*measurable.*

Proof. The asserted measurability properties of $\mathrm{dom}(T^\dagger)$ and $T^\dagger$ follow from Proposition 4.5, which tells us that there is a sequence of continuous operators on ℓ_2 such that $\mathrm{dom}(T^\dagger)$ is its domain of point-wise convergence and $T^\dagger$ is its point-wise limit. □

Remark 7.4. The domain of a generalized inverse $T^\dagger$ is always a measurable linear space, as Lemma 7.3 shows. So if $\gamma \in \mathcal{G}_c$, then $\gamma(\mathrm{dom}(T^\dagger))$ is either 0 or 1 by [Bog98, Theorem 2.5.5]. A reasonable definition of the average-case computability of $T^\dagger$ is only possible in the latter case. In this case, we extend the notation introduced in Subsection 5.1.3 and denote by $[T^\dagger]_\gamma$ the unique γ-equivalence class that contains an extension of $T^\dagger$ to ℓ_2. ($[T^\dagger]_\gamma$ then of course contains all such extensions.) From a practical point of view, there is no distinction between $T^\dagger$ and any element of $[T^\dagger]_\gamma$.

Lemma 7.5. *Suppose* $T \in B(\ell_2)$. *Let* ν *be a Borel measure on* ℓ_2 *with* $\nu(\mathrm{dom}(T^\dagger)) = 1$. *Then* $T^\dagger$ *allows a* $(\mathcal{B}(\ell_2)_\nu, \mathcal{B}(\ell_2))$-*measurable linear extension to* ℓ_2.

Proof. Choose any Hamel basis H_0 of $\mathrm{dom}(T^\dagger)$ and extend it to a Hamel basis H_1 of ℓ_2. Choose a linear mapping $\widetilde{T^\dagger}$ that is equal to $T^\dagger$ on H_0 (and takes arbitrary values on $H_1 \setminus H_0$). $\widetilde{T^\dagger}$ is $(\mathcal{B}(\ell_2)_\nu, \mathcal{B}(\ell_2))$-measurable by Lemmas 7.3 and 5.1. □

Proposition 7.6. *Suppose* $T \in B(\ell_2)$, $\gamma \in \mathcal{G}_c$, $\gamma(\mathrm{dom}(T^\dagger)) = 1$, *and* $(b_n)_n \in \mathrm{ONB}(\gamma)$ *(see Lemma 6.15). Let* F *be any* $(\mathcal{B}(\ell_2)_\gamma, \mathcal{B}(\ell_2))$*-measurable linear extension of* $T^\dagger$ *(see Lemma 7.5). Then, for every* $n \in \mathbb{N}$,

$$\left\|F - \sum_{i=0}^{n} \langle b_i \mid \cdot \rangle\, T^\dagger R_\gamma b_i\right\|_{2,\gamma}^2 = \|F\|_{2,\gamma}^2 - \sum_{i=0}^{n} \|T^\dagger R_\gamma b_i\|^2 \overset{n\to\infty}{\longrightarrow} 0. \tag{7.2}$$

Proof. If we replace $T^\dagger$ by F in (7.2), the claims follow directly from Corollary 6.8 and Lemma 6.10. We have, however, $T^\dagger R_\gamma b_n = F R_\gamma b_n$ for every n because every $R_\gamma b_n$ is in $\mathrm{dom}(T^\dagger)$: In fact, the $R_\gamma b_n$ are in the Cameron-Martin space $H(\gamma)$, and we have $H(\gamma) \subseteq \mathrm{dom}(T^\dagger)$ because $H(\gamma)$ is equal to the intersection of all full-measure linear subspaces of ℓ_2 (see [Bog98, Theorem 2.4.7]). □

7.2.2 Σ_2^0-hardness

We believe that MEAN-computability as considered in Chapter 5 is a natural analogon of IBC's *solvability in the average-case* in computable analysis. We will see, however, that even with the weaker APP/AE-computability, which corresponds to IBC's *probabilistic setting* (see [TWW88]), we get a Σ_2^0-hardness result.

Let $(\ell_2, \|\cdot\|, e)$ be the canonical infinite dimensional computable Hilbert space over $\mathbb{R}$.

An operator $T \in B(\ell_2)$ is **diagonal** if there is a sequence $(x_i)_i \in \mathbb{R}^\omega$ such that

$$(\forall\, i \in \mathbb{N})\ (Te(i) = x_i e(i)).$$

In this case we also write $\mathrm{diag}(x_0, x_1, \ldots)$ for this operator. $\mathrm{diag}(x_0, x_1, \ldots)$ is injective if, and only if, all x_i are non-zero. In this case, the range of $\mathrm{diag}(x_0, x_1, \ldots)$ is dense in ℓ_2 and thus its generalized inverse is equal to its inverse.

Theorem 7.7. *Let* γ *be the centered Gaussian measure on* ℓ_2 *with*

$$R_\gamma = \mathrm{diag}(1, 2^{-1}, 2^{-2}, \ldots).$$

Consider the mapping

$$\mathrm{INVDIAG} :\subseteq B(\ell_2) \to \Lambda(\alpha^e_{\mathrm{Cauchy}} \to \alpha^e)^\gamma_{\mathrm{APP/AE}}$$

with

$$\mathrm{dom}(\mathrm{INVDIAG}) := \{T \ :\ T \textit{ diagonal and injective},\ \gamma(\mathrm{dom}(T^{-1})) = 1\}$$

and

$$\mathrm{INVDIAG}(T) := [T^{-1}]_\gamma.$$

$(\mathrm{INVDIAG}; \Delta, [\alpha^e_{\mathrm{Cauchy}} \to \alpha^e]^\gamma_{\mathrm{APP/AE}})$ *is* Σ_2^0*-hard.*

The proof requires some preparation:

Proposition 7.8. *Let ν be a $\theta^e_{\mathcal{M}_0<}$-computable Borel probability measure on ℓ_2. Consider the mapping*

$$\mathrm{IMGTRACE} :\subseteq \Lambda(\alpha^e_{\mathrm{Cauchy}} \to \alpha^e)^{\nu}_{\mathrm{APP/AE}} \to \mathbb{R}$$

with

$$\mathrm{dom}(\mathrm{IMGTRACE}) := \{[F]_\gamma \;:\; \nu \circ F^{-1} \textit{ is centered Gaussian}\}$$

and

$$\mathrm{IMGTRACE}([F]_\nu) := \int \|F\|^2 \, d\nu.$$

$\mathrm{IMGTRACE}$ *is* $([\alpha^e_{\mathrm{Cauchy}} \to \alpha^e]^{\nu}_{\mathrm{APP/AE}}, \rho_{\mathbb{R}})$*-computable.*

Proof. Proposition 2.28 yields that ν is even $\theta^e_{\mathcal{M}_0}$-computable. By Theorem 5.23, we can convert the $[\alpha^e_{\mathrm{Cauchy}} \to \alpha^e]^{\gamma}_{\mathrm{APP/AE}}$-name of $[F]_\gamma$ into an $[\alpha^e_{\mathrm{Cauchy}} \to \alpha^e]^{\gamma}_{\mathrm{APP}}$-name. We thus have the necessary input information to invoke Theorem 5.32 and compute a $\theta^e_{\mathcal{M}_0<}$-name of $\nu \circ F^{-1}$. As $\nu \circ F^{-1}$ is assumed to be centered Gaussian, we have in fact a Γ_{top}-name of this measure. By Lemma 6.20, we can $\rho_{\mathbb{R}}$-compute $\mathrm{trace}(R_{\nu\circ F^{-1}})$. Finally, recall that by Lemma 6.12

$$\mathrm{trace}(R_{\nu\circ F^{-1}}) = \int \|\cdot\|^2 \, d(\nu \circ F^{-1}) = \int \|F\|^2 \, d\nu.$$

□

Lemma 7.9. *Consider the mapping*

$$A :\subseteq \mathbb{R}^\omega \to \mathbb{R}$$

given by the condition that $A((a_i)_i) = x$ if, and only if,

(1) $0 < a_i < 1$ *for every* $i \in \mathbb{N}$,

(2) $(a_i)_i$ *is nondecreasing,*

(3) $(a_i)_i$ *converges to* x.

$(A; [\rho_{\mathbb{R}}]^\omega, \rho_{\mathbb{R}})$ *is* Σ^0_2*-hard.*

Proof. We reduce $(C_1; \mathrm{id}_{\mathbb{N}^\omega}, \mathrm{id}_{\mathbb{N}^\omega})$ to $(A; [\rho_{\mathbb{R}}]^\omega, \rho_{\mathbb{R}})$. This is done via a standard argument similar to [Wei00, Example 1.3.2]. We sketch the proof:

Preprocessing: Let input $p \in \mathbb{N}^\omega$ be given. Put

$$D_p := \{n \in \mathbb{N} \;:\; C_1(p)(n) = 0\}.$$

It is easy to see that we can compute a sequence $(a'_i)_i$ of nonnegative numbers that converges nondecreasingly to

$$x' := \sum_{n \in D_p} 3^{-(n+2)}.$$

Put $a_i := a'_i + 3^{-1}$. Then $(a_i)_i$ fulfills property (1) and converges to $x = x' + 3^{-1}$. Pass $(a_i)_i$ to A.

Postprocessing: A gives us a $\rho_{\mathbb{R}}$-name of x. The unique ternary expansion of x can be computed from that name. This gives us the characteristic function of D_p, which is all we need to compute $C_1(p)$. □

Lemma 7.10. *Consider the mapping*

$$B :\subseteq \mathbb{R}^\omega \times \mathbb{N} \to \mathbb{R}$$

given by the condition that $B((x_i)_i, N) = x$ *if, and only if,*

$$(\forall\, i \in \mathbb{N})\ (0 < x_i \leq N) \tag{7.3}$$

and

$$\sum_{i=0}^{\infty} 2^{-i} x_i^{-2} = x. \tag{7.4}$$

$(B; [\rho_{\mathbb{R}}]^\omega, \rho_{\mathbb{R}})$ *is* Σ_2^0*-hard.*

Proof. We prove the claim by reducing the previous lemma's $(A; [\rho_{\mathbb{R}}]^\omega, \rho_{\mathbb{R}})$ to $(B; [\rho_{\mathbb{R}}]^\omega, \rho_{\mathbb{R}})$. The reduction will only use preprocessing; the postprocessing will simply forward the output of B. So we need to demonstrate how to compute, from any given $(a_i)_i \in \mathrm{dom}(A)$, a sequence $(x_i)_i$ of positive numbers with

$$\sum_{i=0}^{\infty} 2^{-i} x_i^{-2} = \lim_{i \to \infty} a_i, \tag{7.5}$$

as well as an upper bound $N \in \mathbb{N}$ of $(x_i)_i$. First find an $m \in \mathbb{N}$ such that $a_0 > 2^{-m}$. For every i, put $b_i := a_i - 2^{-(m+i)}$. Obviously,

$$\lim_{i \to \infty} a_i = \lim_{i \to \infty} b_i$$

and

$$b_i - b_{i-1} = \underbrace{a_i - a_{i-1}}_{\geq 0} - 2^{-(m+i)} + 2^{-(m+i-1)} \geq 2^{-(m+i)}. \tag{7.6}$$

Put $x_0 = b_0^{-1/2}$ and

$$x_i = (b_i - b_{i-1})^{-1/2} 2^{-i/2}$$

for $i \geq 1$. The estimate (7.6) yields $0 < x_i \leq 2^{m/2}$ for $i \geq 1$. We may hence choose $N = \lceil \max(x_0, 2^{m/2}) \rceil$. Furthermore, an elementary induction shows

$$\sum_{i=0}^{n} 2^{-i} x_i^{-2} = b_n$$

for every n, and (7.5) follows. □

Proof of Theorem 7.7. We prove the Σ_2^0-hardness by reducing Lemma 7.10's $(B; [\rho_{\mathbb{R}}]^\omega, \rho_{\mathbb{R}})$ to $(\mathrm{INVDIAG}; \Delta, [\alpha^e_{\text{Cauchy}} \to \alpha^e]^\gamma_{\text{APP/AE}})$. So let $((x_i)_i, N) \in \mathrm{dom}(B)$ be given in the required encoding.

Preprocessing: The sequence $(x_i)_i$ defines an operator

$$\mathrm{diag}(x_0, x_1, \ldots) =: T \in B(\ell_2).$$

In fact, we have

$$\|T(\alpha_0, \alpha_1, \ldots)\| = \left(\sum_{i=0}^{n} \alpha_i^2 x_i^2 \right)^{1/2} \leq N \|(\alpha_0, \alpha_1, \ldots)\|$$

for every $(\alpha_0, \alpha_1, \ldots) \in \ell_2$, so $\|T\| \leq N$. This means that we have all the information we need to compute a $\delta^{e,e}_{\text{seq},\geq}$-name and hence a $\delta^{e,e}_{\text{ev}}$-name of T. As T is self-adjoint, we can trivially even compute a Δ-name of T. By assumption, all x_i are non-zero, so T is injective. It is easy to see that

$$\mathrm{range}(T) = \left\{ a \in \ell_2 \ : \ \sum_{i=0}^{\infty} x_i^{-2} \langle e(i) \mid a \rangle^2 < \infty \right\}.$$

Note that

$$\begin{aligned} \int \sum_{i=0}^{\infty} x_i^{-2} \langle e(i) \mid a \rangle^2 \gamma(da) &= \sum_{i=0}^{\infty} x_i^{-2} \int \langle e(i) \mid a \rangle^2 \gamma(da) = \sum_{i=0}^{\infty} x_i^{-2} \langle R_\gamma e(i) \mid e(i) \rangle \\ &= \sum_{i=0}^{\infty} x_i^{-2} 2^{-i} < \infty, \end{aligned}$$

so the integrand on the left-hand side must be finite γ-a.e., which means $\gamma(\mathrm{range}(T)) = 1$. We thus have verified $T \in \mathrm{dom}(\mathrm{INVDIAG})$. Pass T to INVDIAG.

Postprocessing: INVDIAG gives us an $[\alpha^e_{\text{Cauchy}} \to \alpha^e]^\gamma_{\text{APP/AE}}$-name of $[T^{-1}]_\gamma = [T^\dagger]_\gamma$. By Lemma 7.5, $[T^{-1}]_\gamma$ contains a $(\mathcal{B}(\ell_2)_\gamma, \mathcal{B}(\ell_2))$-measurable linear extension F of T^{-1}. The measure $\gamma \circ F^{-1}$ is centered Gaussian by Lemma 6.2. We can hence apply the computable mapping IMGTRACE from Proposition 7.8 and $\rho_{\mathbb{R}}$-compute the number $\|F\|^2_{2,\gamma}$. It is now sufficient to show

$$\|F\|^2_{2,\gamma} = B((x_i)_i, N) = \sum_{i=0}^{\infty} 2^{-i} x_i^{-2}.$$

We have already seen in the proof of Lemma 6.15 that $(\langle e(i) \mid \cdot \rangle)_i$ is complete in $(\ell_2)^*_\gamma$. The sequence $(\langle 2^{i/2} e(i) \mid \cdot \rangle)_i$ is still complete in $(\ell_2)^*_\gamma$, but in addition orthonormal, i.e. $(2^{i/2} e(i))_i \in \mathrm{ONB}(\gamma)$. Proposition 7.6 yields

$$\|F\|^2_{2,\gamma} = \sum_{i=0}^{\infty} \|T^{-1} R_\gamma(2^{i/2} e(i))\|^2 = \sum_{i=0}^{\infty} \|T^{-1}(2^{-i/2} e(i))\|^2 = \sum_{i=0}^{\infty} 2^{-i} x_i^{-2}.$$

□

Let us formulate the following immediate corollary of Theorem 7.7 and Lemma 2.30:

Corollary 7.11. *Let γ be as in Theorem 7.7. There exists a Δ-computable diagonal injective $T \in B(\ell_2)$ such that $\gamma(\mathrm{dom}(T^{-1})) = 1$ and $[T^{-1}]_\gamma$ is not $[\alpha^e_{\mathrm{Cauchy}} \to \alpha^e]^\gamma_{\mathrm{APP/AE}}$-computable.* □

7.2.3 Σ^0_2-computability

Now that we have proved the strongly negative Corollary 7.11, we round out the picture with a Σ^0_2-effective version of Proposition 7.6.

Let $\mathrm{FR} \subseteq B(\ell_2)$ be the set of all finite rank operators of the form

$$x \mapsto \sum_{i=1}^{n} \langle a_i \mid x \rangle \, b_i.$$

We introduce the representation δ_{FR} of FR:

$$\delta_{\mathrm{FR}}\langle r, s\rangle_{\omega\omega} = \Big(x \mapsto \sum_{i=1}^{n} \langle a_i \mid x \rangle \, b_i\Big) \quad :\Longleftrightarrow \quad \left[\begin{array}{l} [\alpha^e_{\mathrm{Cauchy}}]^{<\omega}(r) = (a_1, \ldots, a_n), \\ [\alpha^e_{\mathrm{Cauchy}}]^{<\omega}(s) = (b_1, \ldots, b_n). \end{array}\right.$$

Consider the partial multi-mapping

$$\mathrm{GI}_{\mathrm{avg}} :\subseteq B(\ell_2) \times \mathcal{G}_c \to \mathrm{FR}^\omega$$

defined by the condition that $\mathrm{GI}_{\mathrm{avg}}(T, \gamma) \ni (\Psi_n)_n$ if, and only if,

(1) $\gamma(\mathrm{dom}(T^\dagger)) = 1$, and

(2) $\|F - \Psi_i\|_{2,\gamma} \leq 2^{-i}$ for every $i \in \mathbb{N}$ and one – and hence all – $F \in [T^\dagger]_\gamma$.

Proposition 7.6 tells us that

$$\mathrm{dom}(\mathrm{GI}_{\mathrm{avg}}) = \{(T, \gamma) \;:\; \gamma(\mathrm{dom}(T^\dagger)) = 1\}.$$

One should first ask whether $\mathrm{GI}_{\mathrm{avg}}$ is $([\Delta, \Gamma_{\mathrm{alg}}], [\delta_{\mathrm{FR}}]^\omega)$-computable, $([\delta^{e,e}_{\mathrm{ev}}, \Gamma_{\mathrm{weak}}], [\delta_{\mathrm{FR}}]^\omega)$-computable, or something in between. This is all not the case because $\mathrm{GI}_{\mathrm{avg}}$ is already at least as hard as INVDIAG from Theorem 7.7:

Proposition 7.12. *Let* γ *and* $\mathrm{INVDIAG}$ *be as in Theorem 7.7. Let* $\mathrm{GI}^{\gamma}_{\mathrm{avg}} \subseteq B(\ell_2) \to \mathrm{FR}^{\omega}$ *be given by*

$$\mathrm{dom}(\mathrm{GI}^{\gamma}_{\mathrm{avg}}) := \{T \ : \ (T,\gamma) \in \mathrm{dom}(\mathrm{GI}_{\mathrm{avg}})\} \quad \textit{and} \quad \mathrm{GI}^{\gamma}_{\mathrm{avg}}(T) := \mathrm{GI}_{\mathrm{avg}}(T,\gamma).$$

Then $(\mathrm{INVDIAG}; \Delta, [\alpha^{e}_{\mathrm{Cauchy}} \to \alpha^{e}]^{\gamma}_{\mathrm{APP/AE}}) \leq_c (\mathrm{GI}^{\gamma}_{\mathrm{avg}}; \Delta, [\delta_{\mathrm{FR}}]^{\omega})$.

Proof. Let a Δ-name of a mapping $T \in \mathrm{dom}(\mathrm{INVDIAG})$ be given. Then also $T \in \mathrm{dom}(\mathrm{GI}^{\gamma}_{\mathrm{avg}})$. Without any special preprocessing, directly apply $\mathrm{GI}^{\gamma}_{\mathrm{avg}}$ to T; let $(\Psi_i)_i$ be the result. In the postprocessing, we need to compute an $(\alpha^{e}_{\mathrm{Cauchy}}, \alpha^{e})^{\gamma}_{\mathrm{APP/AE}}$-realization of an element F of $[T^{\dagger}]_{\gamma}$; we show that we can even compute an $(\alpha^{e}_{\mathrm{Cauchy}}, \alpha^{e})^{\gamma}_{\mathrm{MEAN}}$-realization Φ of F (cf. Proposition 5.19.3). For any given $p \in \mathrm{dom}(\alpha^{e}_{\mathrm{Cauchy}})$ and $n \in \mathbb{N}$, compute $\phi(p,n)$ to be an α^{e}-name of a 2^{n+1}-approximation to $\Psi_{2^{n+1}}$ – it is easy to see that the available δ_{FR}-name of $\Psi_{2^{n+1}}$ provides enough information to perform such a computation. Then

$$\begin{aligned}
\int^{*} e(F, \alpha^{e}_{\mathrm{Cauchy}}, \Phi_n, x)\,\gamma(dx) &\leq \int^{*} e(\Psi_{2^{n+1}}, \alpha^{e}_{\mathrm{Cauchy}}, \Phi_n, x)\,\gamma(dx) + \|F - \Psi_{2^{n+1}}\|_{1,\gamma} \\
&\leq 2^{n+1} + \|F - \Psi_{2^{n+1}}\|_{1,\gamma} \\
&\leq 2^{n+1} + \|F - \Psi_{2^{n+1}}\|_{2,\gamma} \\
&\leq 2^{n}.
\end{aligned}$$

(The third estimate is Lyapunov's inequality; see [Shi96].) So Φ is a $(\alpha^{e}_{\mathrm{Cauchy}}, \alpha^{e})^{\gamma}_{\mathrm{MEAN}}$-realization of F. □

Theorem 7.13. $(\mathrm{GI}_{\mathrm{avg}}; [\Delta, \Gamma_{\mathrm{weak}}], [\delta_{\mathrm{FR}}]^{\omega})$ *is* Σ^0_2*-computable.*

Proof. We prove the Σ^0_2-computability by reduction to LIM (see Proposition 2.32). Let a $[\Delta, \Gamma_{\mathrm{weak}}]$-name of some $(T,\gamma) \in \mathrm{dom}(\mathrm{GI}_{\mathrm{avg}})$ be given. Let $F \in [T^{\dagger}]_{\gamma}$ be an $(\mathcal{B}(\ell_2)_{\gamma}, \mathcal{B}(\ell_2))$-measurable linear extension of $T^{\dagger}$ (see Lemma 7.5). It is sufficient to $[\delta_{\mathrm{FR}}]^{\omega}$-compute a sequence $(\Psi_i)_i \in \mathrm{FR}^{\omega}$ such that $\|F - \Psi_i\|_{2,\gamma} \leq 2^{-i}$ for every $i \in \mathbb{N}$.

Preprocessing: By Lemma 6.15, we can $[\alpha^{e}_{\mathrm{Cauchy}}]^{\omega}$-compute a sequence $(b_n)_n$ such that $(\langle b_n \mid \cdot \rangle)_n$ is an orthonormal basis of $(\ell_2)^{*}_{\gamma}$. Note that (7.2) holds. Compute $\mathrm{TYKH}(T) =: (F_k)_k$ (see Proposition 4.3). By the properties of TYKH, we have, for every n and every k,

$$\begin{aligned}
&\Big\| \sum_{i=0}^{n} \langle b_i \mid \cdot \rangle\, T^{\dagger} R_{\gamma} b_i - \sum_{i=0}^{n} \langle b_i \mid \cdot \rangle\, F_k T^{*} R_{\gamma} b_i \Big\|^2_{\gamma,2} \\
&= \sum_{i=0}^{n} \|T^{\dagger} R_{\gamma} b_i - F_k T^{*} R_{\gamma} b_i\|^2 \qquad (7.7)\\
&\leq \sum_{i=0}^{n} \|T^{\dagger} R_{\gamma} b_i\|^2 - \sum_{i=0}^{n} \|F_k T^{*} R_{\gamma} b_i\|^2 \stackrel{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

We also have, for every n, k, and ℓ,

$$\begin{aligned}
&\Big\| \sum_{i=0}^{n} \langle b_i \mid \cdot \rangle \, F_k T^* R_\gamma b_i - \sum_{i=0}^{n} \langle b_i \mid \cdot \rangle \, \sum_{j=0}^{\ell} \langle F_k T^* R_\gamma b_i \mid e(j) \rangle e(j) \Big\|_{\gamma,2}^2 \\
&= \sum_{i=0}^{n} \Big\| F_k T^* R_\gamma b_i - \sum_{j=0}^{\ell} \langle F_k T^* R_\gamma b_i \mid e(j) \rangle e(j) \Big\|^2 \\
&= \sum_{i=0}^{n} \| F_k T^* R_\gamma b_i \|^2 - \sum_{i=0}^{n} \sum_{j=0}^{\ell} \langle F_k T^* R_\gamma b_i \mid e(j) \rangle^2 \overset{\ell \to \infty}{\longrightarrow} 0.
\end{aligned} \tag{7.8}$$

For abbreviation, let us put

$$h_i := T^\dagger R_\gamma b_i, \qquad h_{i,k} := F_k T^* R_\gamma b_i \quad \text{and} \quad h_{i,k,\ell} := \sum_{j=0}^{\ell} \langle F_k T^* R_\gamma b_i \mid e(j) \rangle e(j).$$

In combination, the convergence statements in (7.2), (7.7), and (7.8) yield

$$\lim_{n\to\infty} \lim_{k\to\infty} \lim_{\ell\to\infty} \Big\| F - \sum_{i=0}^{n} \langle b_i \mid \cdot \rangle \, h_{i,k,\ell} \Big\|_{2,\gamma} = 0.$$

In particular

$$\lim_{n\to\infty} \lim_{k\to\infty} \lim_{\ell\to\infty} \Big\| \sum_{i=0}^{n} \langle b_i \mid \cdot \rangle \, h_{i,k,\ell} \Big\|_{2,\gamma} = \|F\|_{2,\gamma} \tag{7.9}$$

Furthermore, adding (7.2), (7.7), and (7.8) yields

$$\begin{aligned}
&\Big\| F - \sum_{i=0}^{n} \langle b_i \mid \cdot \rangle \, h_i \Big\|_{2,\gamma}^2 + \Big\| \sum_{i=0}^{n} \langle b_i \mid \cdot \rangle \, h_i - \sum_{i=0}^{n} \langle b_i \mid \cdot \rangle \, h_{i,k} \Big\|_{\gamma,2}^2 \\
&\qquad + \Big\| \sum_{i=0}^{n} \langle b_i \mid \cdot \rangle \, h_{i,k} - \sum_{i=0}^{n} \langle b_i \mid \cdot \rangle \, h_{i,k,\ell} \Big\|_{\gamma,2}^2 \\
&\leq \|F\|_{2,\gamma}^2 - \sum_{i=0}^{n} \|h_{i,k,\ell}\|^2.
\end{aligned}$$

Via the Cauchy-Schwarz inequality[4], this yields

$$\begin{aligned}
&\big\|F - \sum_{i=0}^{n} \langle b_i \mid \cdot\rangle\, h_{i,k,\ell}\|_{2,\gamma} \\
&\le \big\|F - \sum_{i=0}^{n} \langle b_i \mid \cdot\rangle\, h_i\big\|_{2,\gamma} + \big\|\sum_{i=0}^{n} \langle b_i \mid \cdot\rangle\, h_i - \sum_{i=0}^{n} \langle b_i \mid \cdot\rangle\, h_{i,k}\big\|_{\gamma,2} \\
&\qquad\qquad + \big\|\sum_{i=0}^{n} \langle b_i \mid \cdot\rangle\, h_{i,k} - \sum_{i=0}^{n} \langle b_i \mid \cdot\rangle\, h_{i,k,\ell}\big\|_{\gamma,2} \\
&\le \sqrt{3}\big(\big\|F - \sum_{i=0}^{n} \langle b_i \mid \cdot\rangle\, h_i\big\|_{2,\gamma}^2 + \big\|\sum_{i=0}^{n} \langle b_i \mid \cdot\rangle\, h_i b_i - \sum_{i=0}^{n} \langle b_i \mid \cdot\rangle\, h_{i,k}\big\|_{\gamma,2}^2 \\
&\qquad\qquad + \big\|\sum_{i=0}^{n} \langle b_i \mid \cdot\rangle\, h_{i,k} - \sum_{i=0}^{n} \langle b_i \mid \cdot\rangle\, h_{i,k,\ell}\big\|_{\gamma,2}^2\big)^{1/2} \\
&\le \sqrt{3}\left(\|F\|_{2,\gamma}^2 - \sum_{i=0}^{n} \|h_{i,k,\ell}\|^2\right)^{1/2}.
\end{aligned} \tag{7.10}$$

In particular

$$\sum_{i=0}^{n} \|h_{i,k,\ell}\| \le \|F\|_{2,\gamma}. \tag{7.11}$$

We can compute $h_{i,k,\ell}$ uniformly in i, k, ℓ; this becomes obvious if one recalls that the F_k are self-adjoint and so

$$h_{i,k,\ell} = \sum_{j=0}^{\ell} \langle F_k T^* R_\gamma b_i \mid e(j)\rangle e(j) = \sum_{j=0}^{\ell} \langle R_\gamma b_i \mid T F_k e(j)\rangle e(j).$$

It is not hard to see that we can compute a sequence $(n_m, k_m, \ell_m)_m$ such that

$$\lim_{m\to\infty} \sum_{i=0}^{n_m} \|h_{i,k_m,\ell_m}\| = \sup_{n,k,\ell} \sum_{i=0}^{n} \|h_{i,k,\ell}\|,$$

which, by (7.9) and (7.11), is equal to $\|F\|_{2,\gamma}$. Pass this sequence to LIM.

Postprocessing: LIM gives us a $\rho_{\mathbb{R}}$-name of $\|F\|_{2,\gamma}$. We can then effectively find, for any given $m \in \mathbb{N}$, numbers $\hat{n}_m, \hat{k}_m, \hat{\ell}_m$ such that

$$\sqrt{3}\left(\|F\|_{2,\gamma}^2 - \sum_{i=0}^{\hat{n}_m} \|h_{i,\hat{k}_m,\hat{\ell}_m}\|^2\right)^{1/2} < 2^{-m}.$$

[4] Here used in the form $a + b + c \le \sqrt{3}\sqrt{a^2 + b^2 + c^2}$, $a, b, c \ge 0$.

Put

$$\Psi_m := \sum_{i=0}^{\hat{n}_m} \langle b_i \mid \cdot \rangle \, h_{i,\hat{k}_m,\hat{\ell}_m}.$$

We have $\|F - \Psi_m\|_{2,\gamma} < 2^{-m}$ by (7.10). $\square$

If the information that is obtained by the call to the LIM operation in the proof of Theorem 7.13 is instead available as additional input, the same proof yields the following computability result:

Corollary 7.14. *Consider the partial multi-mapping*

$$\mathrm{GI}^+_{\mathrm{avg}} :\subseteq B(\ell_2) \times \mathcal{G}_c \times \mathbb{R} \to \mathrm{FR}^\omega$$

given by the condition that $\mathrm{GI}^+_{\mathrm{avg}}(T, \gamma, c) \ni (\Psi_n)_n$ *if, and only if,* $\mathrm{GI}_{\mathrm{avg}}(T, \gamma) \ni (\Psi_n)_n$ *and*

$$c = \int_{\mathrm{dom}(T^\dagger)} \|T^\dagger\|^2 \, d\gamma.$$

$\mathrm{GI}^+_{\mathrm{avg}}$ *is* $([\Delta, \Gamma_{\mathrm{weak}}, \rho_{\mathbb{R}>}], [\delta_{\mathrm{FR}}]^\omega)$*-computable.* $\square$

Combining Theorem 7.7, Proposition 7.12, and Theorem 7.13, we have the following picture:

Corollary 7.15. $(\mathrm{GI}_{\mathrm{avg}}; [\Delta, \Gamma_{\mathrm{weak}}], [\delta_{\mathrm{FR}}]^\omega)$ *and* $(\mathrm{INVDIAG}; \Delta, [\alpha^e_{\mathrm{Cauchy}} \to \alpha^e]^\gamma_{\mathrm{APP/AE}})$ *are* Σ^0_2*-complete.* $\square$

7.3 Conclusion and further directions

In this section we gave three different answers to Traub and Werschulz' question: Proposition 7.2, which is positive but non-uniform; Corollary 7.11, which is negative; and Corollary 7.14, which is positive, but assumes that additional information is available.

In order to assess the practical relevance of Corollary 7.14, one would have to find out whether the critical number

$$\int_{\mathrm{dom}(T^\dagger)} \|T^\dagger\|^2 \, d\gamma$$

is available in applications in which inversion problems with Gaussian-distributed problem elements come up. We have not learned of such applications, yet.

We have so far only characterized the average-case uncomputability of unbounded linear operators that are given as inverses of bounded operators whose adjoints are also known. It remains an open question whether the uncomputability becomes worse if the direct operator's adjoint is *not* available. Even more generally, one could study how uncomputable on the average an unbounded linear operator (of Banach spaces) that fulfills the effectivity conditions of the First Main Theorem can possibly be.

Bibliography

[Ban32] S. Banach. *Théorie des opérations linéaires*. Monografje Matematyczne, Warsaw, 1932.

[BB85] E. Bishop and D.S. Bridges. *Constructive Analysis*. Springer, Berlin, 1985.

[BC06] M. Braverman and S. Cook. Computing over the reals: Foundations for scientific computing. *Notices Amer. Math. Soc.*, 53(3), 2006.

[BCSS98] L. Blum, F. Cucker, M. Shub, and S. Smale. *Complexity and Real Computation*. Springer, New York, 1998.

[BD05] V. Brattka and R. Dillhage. Computability of the spectrum of self-adjoint operators. *J.UCS*, 11(12):1884–1900, 2005.

[BD07] V. Brattka and R. Dillhage. Computability of compact operators on computable Banach spaces with bases. *Math. Logic Quart.*, 53(4/5):345–364, 2007.

[BHW08] V. Brattka, P. Hertling, and K. Weihrauch. A tutorial on computable analysis. In S.B. Cooper, B. Löwe, and A. Sorbi, editors, *New Computational Paradigms: Changing Conceptions of What is Computable*, pages 425–491. Springer, New York, 2008.

[Bog98] V.I. Bogachev. *Gaussian Measures*. AMS, Providence, 1998.

[Bos07] V. Bosserhoff. Computability of solutions of operator equations. *Math. Logic Quart.*, 53:326–344, 2007.

[Bos08a] V. Bosserhoff. Are unbounded linear operators computable on the average for Gaussian measures? *J. Complexity*, 24(4):477–491, 2008.

[Bos08b] V. Bosserhoff. The bit-complexity of finding nearly optimal quadrature rules for weighted integration. *J.UCS*, 14(6):938–955, 2008.

[Bos08c] V. Bosserhoff. Notions of probabilistic computability on represented spaces. *J.UCS*, 14(6):956–995, 2008.

[Bos08d] V. Bosserhoff. On the effective existence of Schauder bases. Preprint, 2008.

[BP03] V. Brattka and G. Presser. Computability on subsets of metric spaces. *Theoret. Comp. Sci.*, 305:43–76, 2003.

[Bra] V. Brattka. A computable version of Banach's inverse mapping theorem. *Ann. Pure Appl. Logic*. (accepted for publication).

[Bra99] V. Brattka. Computable invariance. *Theoret. Comput. Sci.*, 210(1):3–20, 1999.

[Bra01] V. Brattka. Computability of Banach space principles. Technical Report 286, FernUniversität Hagen, Fachbereich Informatik, June 2001.

[Bra03a] V. Brattka. Effective representations of the space of linear bounded operators. *Appl. Gen. Topol.*, 4(1):115–131, 2003.

[Bra03b] V. Brattka. The emperor's new recursiveness: The epigraph of the exponential function in two models of computability. In M. Ito and T. Imaoka, editors, *Words, Languages & Combinatorics III*, pages 63–72, Singapore, 2003. World Scientific Publishing.

[Bra05] V. Brattka. Effective Borel measurability and reducibility of functions. *Math. Logic Quart.*, 51(1):19–44, 2005.

[Bra08a] V. Brattka. Borel complexity and computability of the Hahn-Banach theorem. *Arch. Math. Logik*, 46(7-8):547–564, 2008.

[Bra08b] V. Brattka. Plottable real number functions and the computable graph theorem. *SIAM J. Comput.*, 38(1):303–328, 2008.

[BSS89] L. Blum, M. Shub, and S. Smale. On a theory of computation and complexity over the real numbers: NP completeness, recursive functions and universal machines. *Bull. Amer. Math. Soc.*, 21:1–46, 1989.

[BV06] D.S. Bridges and L.S. Vita. *Techniques of Constructive Analysis*. Springer, Berlin, 2006.

[BY06] V. Brattka and A. Yoshikawa. Towards computability of elliptic boundary value problems in variational formulation. *J. Complexity*, 22(6):858 – 880, 2006.

[Coh80] D.L. Cohn. *Measure Theory*. Birkhäuser, Boston, 1980.

[Dav73] A.M. Davie. The approximation problem for Banach spaces. *Bull. London Math. Soc.*, 5:261–266, 1973.

[Dil08] R. Dillhage. Computability of the spectrum of self-adjoint operators and the computable operational calculus. *Electron. Notes Theor. Comput. Sci.*, 202:339–364, 2008.

[DK00] D. Du and K. Ko. *Theory of Computational Complexity*. John Wiley & Sons, 2000.

[EHN96] H.W. Engl, M. Hanke, and A. Neubauer. *Regularization of Inverse Problems*. Kluwer, 1996.

[Enf73] P. Enflo. A counterexample to the approximation property in Banach spaces. *Acta Math.*, 130:309–317, 1973.

[Gác05] P. Gács. Uniform test of algorithmic randomness over a general space. *Theoret. Comput. Sci.*, 341:91 – 137, 2005.

[Ghe06] G. Gherardi. Effective Borel degrees of some topological functions. *Math. Logic Quart*, 52(6):625–642, 2006.

[GM08] G. Gherardi and A. Marcone. How much incomputable is the separable Hahn-Banach theorem? Preprint, 2008.

[Gro77] C.W. Groetsch. *Generalized Inverses of Linear Operators*. Dekker, New-York, 1977.

[Gro80] C.W. Groetsch. *Elements of Applicable Functional Analysis*. Dekker, New York, 1980.

[Gro93] C.W. Groetsch. *Inverse Problems in the Mathematical Sciences*. Vieweg, Braunschweig, 1993.

[Grz57] A. Grzegorczyk. On the definition of computable real continuous functions. *Fund. Math.*, 44:61–71, 1957.

[GSW07] T. Grubba, M. Schröder, and K. Weihrauch. Computable metrization. *Math. Logic Quart.*, 53:381–395, 2007.

[GWX08] T. Grubba, K. Weihrauch, and Y. Xu. Effectivity on continuous functions in topological spaces. In R. Dillhage, T. Grubba, A. Sorbi, K. Weihrauch, and N. Zhong, editors, *Proceedings of the Fourth International Conference on Computability and Complexity in Analysis (CCA 2007)*, volume 202 of *Electron. Notes Theor. Comput. Sci.*, pages 237–254. Elsevier, 2008.

[Her05] P. Hertling. On the solvability of ill-posed problems in two models of computation. Talk presented at the Foundations of Computational Mathematics conference, Santander, Spain, July 2005.

[IEE85] IEEE, New York. *IEEE Standard for Binary Floating-Point Arithmetic. ANSI/IEEE Std. 754-1985*, 1985.

[Kal02] O. Kallenberg. *Foundations of Modern Probability*. Springer, New York, second edition, 2002.

[KMP$^+$04] L. Kettner, K. Mehlhorn, S. Pion, S. Schirra, and C. Yap. Classroom examples of robustness problems in geometric computation. In S. Albers and T. Radzik, editors, *Algorithms - ESA 2004, 12th Annual European Symposium, Bergen, Norway, September 14-17, 2004, Proceedings*, volume 3221 of *Lecture Notes in Computer Science*, pages 702–713. Springer, 2004.

[Ko91] K.-I Ko. *Complexity Theory of Real Functions*. Birkhäuser, Boston, 1991.

[KRW91] M.A. Kon, K. Ritter, and A.G. Werschulz. On the average case solvability of ill-posed problems. *J. Complexity*, 7:220–224, 1991.

[KW85] Chr. Kreitz and K. Weihrauch. Theory of representations. *Theoret. Comput. Sci.*, 38:35–53, 1985.

[Lac55] D. Lacombe. Extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelle I-III. *C. R. Acad. Sci. Paris*, 240/241, 1955.

[Meg98] R.E. Megginson. *An Introduction to Banach Space Theory*. Springer, 1998.

[MN82] G. Metakides and A. Nerode. The introduction of non-recursive methods into mathematics. In A. Troelstra and D. v. Dalen, editors, *The L.E.J. Brouwer Centenary Symposium*, volume 110 of *Stud. Logic Found. Math.*, pages 319–335, 1982.

[Nov95] E. Novak. The real number model in numerical analysis. *J. Complexity*, 11:57 – 73, 1995.

[NW99] E. Novak and H. Woźniakowski. On the cost of uniform and nonuniform algorithms. *Theoret. Comput. Sci.*, 219:301–318, 1999.

[Par03] M.W. Parker. Undecidability in $\mathbb{R}^n$: Riddled basins, the KAM tori, and the stability of the solar system. *Philos. Sci.*, 70(2):359–382, 2003.

[Par05] M.W. Parker. *Undecidable long-term behavior in classical physics: Foundations, results, and interpretation*. PhD thesis, University of Chicago, 2005.

[Par06] M.W. Parker. Three concepts of decidability for general subsets of uncountable spaces. *Theoret. Comput. Sci.*, 351:2 – 13, 2006.

[Pen89] R.L. Penrose. *The Emperor's New Mind*. Oxford University Press, 1989.

[PER89] M.B. Pour-El and J.I. Richards. *Computability in Analysis and Physics*. Springer, Berlin, 1989.

[PEZ97] M. Pour-El and N. Zhong. The wave equation with computable initial data whose unique solution is nowhere computable. *Math. Logic Quart.*, 43:499–509, 1997.

[Pla96] L. Plaskota. *Noisy Information and Computational Complexity*. Cambridge University Press, Cambridge, 1996.

[Puj75] L. Pujara. Some local structures in Banach spaces. *Math. Japonicae*, 20:49–54, 1975.

[Rog87] H. Rogers. *Theory of Recursive Functions and Effective Computability*. MIT Press, Cambridge, 1987.

[Sch98] M. Schröder. Effective metrization on regular spaces. In K.-I Ko, A. Nerode, M.B. Pour-El, K. Weihrauch, and J. Wiedermann, editors, *Computability and Complexity in Analysis (Proceedings of CCA Workshop, Brno, Czech Republic, August 1998)*, volume 235 of *Informatik Berichte*, pages 63 – 80, 1998.

[Sch99] H.H. Schaefer. *Topological Vector Spaces*. Springer, 1999.

[Sch02a] M. Schröder. *Admissible Representations for Continuous Computations*. PhD thesis, Fern-Universität Hagen, 2002.

[Sch02b] M. Schröder. Effectivity in spaces with admissible multirepresentations. *Math. Logic Quart.*, 48:78–90, 2002.

[Sch02c] M. Schröder. Extended admissibility. *Theoret. Comput. Sci.*, 284:519 – 538, 2002.

[Sch07] M. Schröder. Admissible representations for probability measures. *Math. Logic Quart.*, 53:431 – 445, 2007.

[Shi96] A.N. Shiryaev. *Probability*. Springer, New-York, second edition, 1996.

[Sin70] I. Singer. *Bases in Banach Spaces I*. Springer, Berlin, 1970.

[Sin81] I. Singer. *Bases in Banach Spaces II*. Springer, Berlin, 1981.

[Sza87] S.J. Szarek. A Banach space without a basis which has the bounded approximation property. *Acta Math.*, 159(1):81–98, 1987.

[Tra99] J.F. Traub. A continuous model of computation. *Phys. Today*, pages 39–43, May 1999.

[Tur36] A.M. Turing. On computable numbers with an application to the “Entscheidungsproblem”. *Proc. London Math. Soc.*, 42(2):230–265, 1936.

[Tur37] A.M. Turing. On computable numbers with an application to the “Entscheidungsproblem”. A correction. *Proc. London Math. Soc.*, 43(2):544–546, 1937.

[TW94] J.F. Traub and A.G. Werschulz. Linear ill-posed problems are solvable on the average for all Gaussian measures. *Math. Intelligencer*, 16(2):42–48, 1994.

[TW98] J.F. Traub and A.G. Werschulz. *Complexity and Information*. Cambridge University Press, New York, 1998.

[TWW88] J.F. Traub, G.W. Wasilkowski, and H. Woźniakowski. *Information-based complexity*. Academic Press, New York, 1988.

[Vak91] N.N. Vakhania. Gaussian mean boundedness of densely defined linear operators. *J. Complexity*, 7(3):225–231, 1991.

[VTC87] N.N. Vakhania, V.I. Tarieladze, and S.A. Chobanian. *Probability Distributions on Banach Spaces*. Kluwer Academic, Dordrecht/Norwell, MA, 1987.

[Wei92] K. Weihrauch. The TTE-interpretation of three hierarchies of omniscience principles. Technical Report 130, FernUniversität Hagen, Fachbereich Informatik, 1992.

[Wei99] K. Weihrauch. Computability on the probability measures on the Borel sets of the unit interval. *Theoret. Comput. Sci.*, 219(1-2):421–437, 1999.

[Wei00] K. Weihrauch. *Computable Analysis: An Introduction*. Springer, Berlin, 2000.

[Wei03] K. Weihrauch. Computational complexity on computable metric spaces. *Math. Logic Quaterly*, 49:3–21, 2003.

[Wer87] A.G. Werschulz. What is the complexity of ill-posed problems? *Numer. Funct. Anal. Optim.*, 9:945–967, 1987.

[Wer91] A.G. Werschulz. *The Computational Complexity of Differential and Integral Equations: An Information-Based Approach*. Oxford University Press, New York, 1991.

[Wil70] S. Willard. *General Topology*. Addison-Wesley, Reading, 1970.

[WZ02] K. Weihrauch and N. Zhong. Is wave propagation computable or can wave computers beat the Turing machine? *Proc. London Math. Soc.*, 85(2):312 – 332, 2002.

[WZ05] K. Weihrauch and N. Zhong. Computing the solution of the Korteweg-de Vries equation with arbitrary precision on turing machines. *Theoret. Comput. Sci.*, 332(1-3):337–366, 2005.

[WZ06a] K. Weihrauch and N. Zhong. An algorithm for computing fundamental solutions. *SIAM J. Comput.*, 35(6):1283–1294, 2006.

[WZ06b] K. Weihrauch and N. Zhong. Computing Schrödinger propagators on type-2 Turing machines. *J. Complexity*, 22(6):918–935, 2006.

[WZ07] K. Weihrauch and N. Zhong. Computable analysis of the abstract Cauchy problem in a Banach space and its applications I. *Math. Logic Quart.*, 53(4-5):511–531, 2007.

[ZB04] M. Ziegler and V. Brattka. Computability in linear algebra. *Theoret. Comput. Sci.*, 326:187–211, 2004.

[ZW03] N. Zhong and K. Weihrauch. Computability theory of generalized functions. *J. ACM*, 50(4):469–505, 2003.

Index