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Boundedness of Iwasawa Invariants of Fine Selmer Groups and Selmer Groups

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Abstract. In this article we consider Selmer groups and fine Selmer groups of abelian varieties over a number field K. Following a classical approach of Monsky for Iwasawa modules from ideal class groups, we give sufficient conditions for the Iwasawa μ-invariants of the fine Selmer groups and the μ-invariants of the Selmer groups to be bounded as one runs over the $\mathbb{Z}_p$-extensions of K. Moreover, we describe a criterion for the boundedness of Iwasawa λ-invariants of Selmer groups and fine Selmer groups over multiple $\mathbb{Z}_p$-extensions which generalises a criterion of Monsky from dimension 2 to arbitrary dimension.

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1. Introduction

Classical Iwasawa theory is concerned with the growth of class numbers in $\mathbb{Z}_p$-extensions of number fields. To be more precise, let K_∞/K be a $\mathbb{Z}_p$-extension, with intermediate fields K_n, $n \in \mathbb{N}$, and let h_n denote the class number of K_n. By a famous result of Iwasawa (see [11]), we have

$$v_p(h_n) = \mu \cdot p^n + \lambda \cdot n + \nu \tag{1}$$

for each sufficiently large $n \in \mathbb{N}$. The so-called *Iwasawa invariants* $\mu \geq 0$, $\lambda \geq 0$ and $\nu \in \mathbb{Z}$ do not depend on n, but they do depend on the chosen $\mathbb{Z}_p$-extension K_∞ of K.

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Having fixed K and p, Greenberg studied in [8] whether the Iwasawa invariants are bounded as one runs over the $\mathbb{Z}_p$-extensions of K, and he was able to obtain first partial results. A few years later, Monsky and independently Babaĭcev (see [1,26]) proved that the μ-invariants of the $\mathbb{Z}_p$-extensions of a fixed number field K are indeed bounded.

On the other hand, the corresponding question for the λ-invariants is still open. In [26, Theorem IV], Monsky obtained a necessary and sufficient criterion for the boundedness of the λ-invariants of the $\mathbb{Z}_p$-extensions of K which are contained in any fixed $\mathbb{Z}_p^2$-extension of K.

It is the main aim of this article to prove analogous results in a more general context. To be more precise, fix K and p, and let A be an abelian variety defined over K. We let $X_A^{(K_\infty)}$ and $Y_A^{(K_\infty)}$ denote the Pontryagin duals of the *Selmer group* and the *fine Selmer group* of A over a $\mathbb{Z}_p$-extension K_∞ of K (see Sect. 2.2 for the definitions). The investigation of the Selmer groups (respectively, the fine Selmer groups) has been introduced into Iwasawa theory by Mazur (respectively, Coates and Sujatha), see [6,24]. This is probably the most vital branch of modern Iwasawa theory.

One can attach μ- and λ-invariants to each $X_A^{(K_\infty)}$ and $Y_A^{(K_\infty)}$. Some technical problems arise since these objects need not be torsion modules over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T]]$. The following theorem is our first main result (see also Theorem 3.20 below).

Theorem 1.1. *Let K be a number field. Suppose that $Y_A^{(K_\infty)}$ is Λ-torsion for all but finitely many $\mathbb{Z}_p$-extensions K_∞ of K. Then $\mu(Y_A^{(K_\infty)})$ is bounded.*

An analogous statement holds for the Selmer groups, provided that A has good ordinary reduction at the primes of K above p, and that the inertia subgroup of $\mathrm{Gal}(\mathbb{L}_\infty/K)$ of every prime of K above p has $\mathbb{Z}_p$-rank at least $d-1$.

In [5], Cuoco and Monsky proved a generalisation of Iwasawa's growth formula (1) for multiple $\mathbb{Z}_p$-extensions: If K_n now denotes the nth layer in a $\mathbb{Z}_p^d$-extension $\mathbb{L}_\infty/K$, then

$$v_p(h_n) = m_0 \cdot p^{dn} + l_0 \cdot np^{n(d-1)} + O(p^{n(d-1)}). \tag{2}$$

We call the leading coefficients $m_0, l_0 \geq 0$ of this asymptotic formula the *generalised Iwasawa invariants* of the $\mathbb{Z}_p^d$-extension $\mathbb{L}_\infty/K$.

In [26, Theorem IV], Monsky obtained the following criterion for the boundedness of λ-invariants: If $\mathbb{L}_\infty/K$ is a $\mathbb{Z}_p^2$-extension, then the λ-invariants of the $\mathbb{Z}_p$-extensions K_∞ of K which are contained in $\mathbb{L}_\infty$ are bounded if and only if $l_0(\mathbb{L}_\infty/K) = 0$.

The definition of the l_0-invariant of an Iwasawa module is a bit complicated (see Sect. 2.2), and its meaning is mysterious. In this article we propose the definition of a slightly modified invariant $\widehat{l_0}$, which coincides with the original l_0-invariant of Cuoco and Monsky in the case of $\mathbb{Z}_p^2$-extensions (more precisely, for Iwasawa modules over the Iwasawa algebra $\Lambda_2 = \mathbb{Z}_p[[T_1, T_2]]$), but may differ from l_0 for $\mathbb{Z}_p^d$-extensions with $d > 2$. Using this invariant $\widehat{l_0}$, we can generalise Monsky's criterion to the setting of $\mathbb{Z}_p^d$-extensions. For the sake of simplicity, we do not state here in the introduction the best possible formulation of our result (see also Theorem 3.22 below):

Theorem 1.2. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, $d \geq 2$, and suppose that $Y_A^{(K_\infty)}$ is Λ-torsion for each $\mathbb{Z}_p$-extension $K_\infty \subseteq \mathbb{L}_\infty$ of K. We assume that the decomposition subgroup in* $\mathrm{Gal}(\mathbb{L}_\infty/K)$ *of every prime of K above p is open, and that $Y_A^{(K_\infty)}$ does not contain any non-trivial pseudo-null Λ_d-submodules. Then $\lambda(Y_A^{(K_\infty)})$ is bounded as one runs over the $\mathbb{Z}_p$-extensions $K_\infty \subseteq \mathbb{L}_\infty$ of K if and only if $\widehat{l_0}(Y_A^{(\mathbb{L}_\infty)}) = 0$.*

An analogous result holds for Selmer groups, provided that A has good ordinary reduction at the primes of K above p, and that each of these primes ramifies in $\mathbb{L}_\infty/K$.

In [18] the first-named author constructed an example of an imaginary-quadratic number field K such that $\lambda(X_A^{(K_\infty)})$ was unbounded as K_∞ runs over the $\mathbb{Z}_p$-extensions of K (here A was a suitable elliptic curve defined over K). Since the composite of all $\mathbb{Z}_p$-extensions of K is a $\mathbb{Z}_p^2$-extension $\mathbb{L}_\infty$ of K, we may use Theorem 1.2 to deduce that $l_0(X_A^{(\mathbb{L}_\infty)}) > 0$. To the authors' knowledge, this yields the first known example of an Iwasawa module with a non-trivial l_0-invariant over an Iwasawa algebra of a $\mathbb{Z}_p^d$-extension with $d > 1$. In the last section, we show how to derive examples of $\mathbb{Z}_p^d$-extensions $\mathbb{L}_\infty$ of K with arbitrarily large d and $\widehat{l_0}(X_A^{(\mathbb{L}_\infty)}) > 0$. (For more information on the relations between the invariants l_0 and $\widehat{l_0}$ we refer the reader to the definitions in Sect. 2.2 and to the end of Sect. 3.3, where we prove several results relating l_0 and $\widehat{l_0}$, see for example Proposition 3.25).

Let us briefly describe the structure of the article. In Sect. 2 we introduce the general background on Iwasawa modules, (generalised) Iwasawa invariants, Selmer groups and fine Selmer groups, and we describe a topology on the set of $\mathbb{Z}_p$-extensions of K which is due to Greenberg. Section 3 contains two preliminary subsections where we prove auxiliary results on Iwasawa modules, and the control theorems which we need. Section 3.3 is devoted to the proofs of our main results.

In Sect. 4, we derive several applications. First, we study the relations between the variation of μ- and λ-invariants, and the weak Leopoldt conjecture. Finally, we construct examples of Iwasawa modules with non-trivial l_0- and $\widehat{l_0}$-invariant.

2. Notation

2.1. General Notation

Fix a number field K and a prime number p, and let Σ be a finite set of primes of K. For any (possibly infinite) algebraic extension N of K, we denote by $\Sigma(N)$ the set of primes w of N which divide some $v \in \Sigma$. Moreover, $\Sigma_p(N)$ will denote the subset of $\Sigma(N)$ which consists of the primes above p (usually our set Σ will contain all the primes of K above p).

We denote by $G_K = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of K, i.e. $\overline{K}$ is a fixed algebraic closure of K. Moreover, K_Σ will denote the maximal algebraic (non-necessarily abelian) extension of K which is unramified outside of Σ.

For any abelian group G, we denote by $G[p^\infty]$ the subgroup of p-power torsion elements of G.

2.2. Group Rings and Iwasawa Modules

For every $\mathbb{Z}_p^d$-extension K_∞/K, $d \in \mathbb{N}$, the completed group ring $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$ can be identified (non-canonically) with the ring

$$\Lambda_d := \mathbb{Z}_p[[T_1, \ldots, T_d]]$$

of formal power series in d variables over $\mathbb{Z}_p$. In [5], Cuoco and Monsky introduced, in this setting, the *generalised Iwasawa invariants* of any finitely generated and torsion Λ_d-module, $d > 1$, as follows: To each such module M, one can attach an *elementary* torsion Λ_d-module of the form

$$E_M = \bigoplus_{i=1}^{s} \Lambda_d/(p^{m_i}) \oplus \bigoplus_{j=1}^{t} \Lambda_d/(g_j^{n_j}),$$

where all the exponents are natural numbers and the g_j are irreducible elements of Λ_d which are coprime with p. Moreover, one has a Λ_d-module homomorphism $\varphi\colon M \longrightarrow E_M$, the kernel and cokernel of which are *pseudo-null* over Λ_d, i.e. annihilated by two relatively prime elements of the unique factorisation domain Λ_d. In fact, the kernel of any such homomorphism φ is the maximal pseudo-null submodule of M, which we denote by M°. The element $F_M = \prod_{i=1}^{s} p^{m_i} \cdot \prod_{j=1}^{t} g_j^{n_j}$ is called the *characteristic power series* of M. It is determined uniquely by M up to multiplication by units of Λ_d. In particular, if $E_M = \{0\}$ (this happens if and only if $M = M^\circ$ is pseudo-null), then we set $F_M = 1$.

One then defines the *generalised Iwasawa invariants* of M as follows. First, $m_0(M) = \sum_{i=1}^{s} m_i$. The definition of the second invariant is more involved. Recall that Λ_d has been identified with the completed group ring $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$. We write $F_M = p^{m_0(M)} \cdot G_M$, and we consider the image $\overline{G_M}$ of G_M in the quotient ring $\Lambda_d/p\Lambda_d$. We let $l_0(M)$ be the sum of the valuations $v_{\mathfrak{p}}(\overline{G_M})$, where $\mathfrak{p}$ runs over the primes of $\Lambda_d/p\Lambda_d$ of the form $\overline{\gamma - 1}$, with $\gamma \in \mathrm{Gal}(K_\infty/K)$ not a pth power (since $\Lambda_d/p\Lambda_d$ is again a unique factorisation domain, this sum is in fact finite). Note that $l_0(M) = \lambda(M)$ if $d = 1$.

In this paper, we will also use an *alternative definition* of the l_0-invariant, which might be more appropriate for Λ_d-modules, $d > 2$, than the classical l_0-invariant (this is motivated by some results in Sect. 3). To this purpose, we assume that $d \geq 2$, and we again write the characteristic power series $F_M \in \Lambda_d$ of a finitely generated Λ_d-module M as $F_M = p^{m_0(M)} \cdot G_M$, i.e. $p \nmid G_M$. Now we define

$$\widehat{l_0}(M) = \sum_{\mathfrak{p}} v_{\mathfrak{p}}(\overline{G_M}),$$

where $\overline{G_M} \in \Omega_d := \Lambda_d / p\Lambda_d$, and where now $\mathfrak{p}$ runs over the prime ideals of Ω_d which are minimal over $(\overline{G_M})$ and contained in a prime ideal of the form

$$\mathcal{P} = (\overline{\sigma_1 - 1}, \ldots, \overline{\sigma_{d-1} - 1}),$$

where $\sigma_1, \ldots, \sigma_{d-1} \in \mathrm{Gal}(K_\infty/K)$ generate a subgroup which is isomorphic to $\mathbb{Z}_p^{d-1}$. Note that this coincides with the original definition of l_0 in the case $d = 2$ since the prime ideals of Ω_2 of the form $(\overline{\gamma - 1})$ have height 1. It is important to restrict to minimal primes in the case $d > 2$ in order to guarantee that the sum $\widehat{l_0}(M)$ is finite.

In the special case of a $\mathbb{Z}_p$-extension, one has a more precise structure theory (see [11]). In this case, we usually abbreviate Λ_1 to Λ. In fact, to any (non-necessarily torsion) finitely generated Λ_1-module M, we can attach an elementary Λ_1-module of the form

$$E_M = \Lambda^r \oplus \bigoplus_{i=1}^{s} \Lambda/(p^{m_i}) \oplus \bigoplus_{j=1}^{t} \Lambda/(g_j^{n_j}),$$

where the g_j are now so-called distinguished polynomials, i.e. they are monic polynomials, and each coefficient besides the leading one is divisible by p. We let $\mu(M) = \sum_{i=1}^{s} m_i$ (i.e. $\mu(M) = m_0(M)$ in the case of finitely generated and torsion Λ_1-modules) and $\lambda(M) = \sum_{j=1}^{t} \deg(g_j^{n_j})$. Using the fact that the g_j are distinguished, it is easy to see that $\lambda(M) = l_0(M)$ for finitely generated torsion Λ_1-modules. Again, one defines the characteristic power series $F_M \in \Lambda$ of M (which now differs from a polynomial only by some unit in $\Lambda^\times$) to be the product of the powers of p and the distinguished polynomials $g_j^{n_j}$ occuring in the definition of E_M.

Using the Weierstraß Preparation Theorem (see [32, Theorem 7.3]), we can define Iwasawa invariants of any non-zero element $f \in \Lambda$, as follows: f can be written uniquely as

$$f = u \cdot p^m \cdot g,$$

where $u \in \Lambda^\times$, $m \in \mathbb{N}$, and $g \in \Lambda$ is a distinguished polynomial. Then we let $\mu(F) := m$ and $\lambda(f) := \deg(g)$. Under this point of view, the Iwasawa invariants of a finitely generated Λ-module M are just the Iwasawa invariants of the associated characteristic power series $F_M \in \Lambda$ of M.

We conclude the current subsection with some *remarks on Fitting ideals*. For a general summary on the properties of Fitting ideals, we refer to [32, Section 13.6], [28, Chapter 3] and [13, Appendix A]. In this article, we will restrict to the *zero-th Fitting ideals* of Iwasawa modules.

Let M be a finitely generated Λ_d-module, $d \in \mathbb{N}$. Recall the definition of the Fitting ideal $\mathcal{F}_{\Lambda_d}(M)$: Take a presentation

$$\Lambda_d^q \xrightarrow{\rho} \Lambda_d^l \longrightarrow M, \tag{3}$$

and let A be a $q \times l$-matrix with entries in Λ_d which describes the map ρ. If M is torsion, then we must have $q \geq l$. Then we let $\mathcal{F}_{\Lambda_d}(M)$ be the ideal of Λ_d generated by all the l-minors of A if M is torsion, and we define it to be the zero ideal if $q < l$. We summarise all the facts on the Fitting ideal which we will need in the following

Proposition 2.1. *Let M be a finitely generated Λ_d-module, and let $\mathrm{Ann}(M) \subseteq \Lambda_d$ denote its annihilator ideal. Then the following statements hold.*

(1) *The definition of $\mathcal{F}_{\Lambda_d}(M)$ does not depend on the choice of the presentation (3) of M.*

(2) *If M can be generated over Λ_d by l elements, then*

$$\mathrm{Ann}(M)^l \subseteq \mathcal{F}_{\Lambda_d}(M) \subseteq \mathrm{Ann}(M).$$

In particular, if M is not a torsion Λ_d-module, then $\mathcal{F}_{\Lambda_d}(M) = (0)$.

(3) *Suppose that M is Λ_d-torsion. Let $M^\circ \subseteq M$ be the maximal pseudo-null Λ_d-submodule, and let E_M be an elementary Λ_d-module which is pseudo-isomorphic to M. If M can be generated over Λ_d by l elements, then*

$$\mathrm{Ann}(M^\circ)\mathrm{Ann}(E_M) \subseteq \mathrm{Ann}(M)$$

and

$$\mathrm{Ann}(M^\circ)^l \mathrm{Ann}(E_M)^l \subseteq \mathcal{F}_{\Lambda_d}(M).$$

(4) *If $i < d$ and $\pi : \Lambda_d \longrightarrow \Lambda_i$ denotes a surjective ring homomorphism, then*

$$\mathcal{F}_{\Lambda_i}(M/\ker(\pi)) = \pi(\mathcal{F}_{\Lambda_d}(M)).$$

(5) *Suppose that M is Λ_d-torsion. Then we can write*

$$\mathcal{F}_{\Lambda_d}(M) = (F_M) \cdot J_M,$$

where the ideal J_M of Λ_d is not contained in any height one prime ideal of Λ_d.

(6) *Now suppose that $d = 1$, and that M is torsion. Then*

$$\mathcal{F}_{\Lambda_1}(M) = (F_M) \cdot \mathcal{F}_{\Lambda_1}(M^\circ).$$

Proof. The first statement follows from [7, Corollary 20.4], and (2) is [7, Proposition 20.7].

The first statement in (3) follows from the exact sequence

$$0 \longrightarrow M^\circ \longrightarrow M \longrightarrow E_M,$$

and the second part of (3) follows by combining the first part with (2).

Assertion (4) follows from [13, Lemma A.6]. Now we turn to the proof of (5). Choose a presentation as in (3) and let A be the corresponding matrix. We will show that the greatest common divisor of the $l \times l$-minors of A is equal to F_M; this proves the assertion.

To this purpose, we will argue prime-by-prime. Let g be an irreducible element of Λ_d (which may or may not divide the characteristic power series F_M of M). In the following, we localise at g. Note that $(\Lambda_d)_{(g)}$ is a flat Λ_d-module and that $N_{(g)} \cong N \otimes_{\Lambda_d} (\Lambda_d)_{(g)}$ for any finitely generated Λ-module N. Therefore we obtain from the exact sequence

$$0 \longrightarrow M^\circ \longrightarrow M \longrightarrow E_M \longrightarrow C \longrightarrow 0,$$

with E_M elementary and C pseudo-null over Λ_d, an exact sequence

$$0 \longrightarrow (M^\circ)_{(g)} \longrightarrow M_{(g)} \longrightarrow (E_M)_{(g)} \longrightarrow C_{(g)}$$

of finitely generated $(\Lambda_d)_{(g)}$-modules. Since both M° and C are pseudo-null over Λ_d, there exists an element h which is coprime with g and annihilates both M° and C. Since h is a unit in $(\Lambda_d)_{(g)}$, we may conclude that both $(M^\circ)_{(g)}$ and $C_{(g)}$ are zero.

We therefore obtain an isomorphism

$$M_{(g)} \cong (E_M)_{(g)}.$$

Note that $(\Lambda_d)_{(g)}$ is a discrete valuation ring with maximal ideal (g). Writing

$$E_M = \bigoplus_{i=1}^{s} \Lambda_d/(f_i)$$

with $f_i \in \Lambda_d$ (here f_i may be equal to p), we obtain that

$$M_{(g)} \cong \bigoplus_{i=1}^{s} (\Lambda_d)_{(g)}/(f_i). \tag{4}$$

In particular, $M_{(g)} = 0$ if each f_i is coprime with g.

On the other hand, it follows from the presentation (3) that we have an exact sequence

$$(\Lambda_d)_{(g)}^q \xrightarrow{\rho} (\Lambda_d)_{(g)}^l \longrightarrow M_{(g)} \longrightarrow 0.$$

Let A be the matrix representing the map ρ. Then $\mathcal{F}_{(\Lambda_d)_{(g)}}(M_{(g)})$ is the ideal generated by the images of the $l \times l$-minors of A under the natural map $\Lambda_d \longrightarrow (\Lambda_d)_{(g)}$.

It follows from [2, Theorem 2.9.6] that the greatest common divisor of the $l \times l$-minors of the matrix A (considered as a matrix over the principal ideal domain $(\Lambda_d)_{(g)}$) is equal to the product of the first l principal divisors of $M_{(g)}$. In view of (4), we may conclude that

$$v_{(g)}(\mathcal{F}_{(\Lambda_d)_{(g)}}) = v_g(F_M).$$

This concludes the proof of (5).

Now suppose that $d = 1$. The theory of Fitting ideals over $\Lambda := \Lambda_1$ is better understood. One of the reasons is that a finitely generated torsion Λ-module M has projective dimension 1 if and only if it does not contain any non-trivial pseudo-null (i.e. finite) Λ-submodules (see e.g. [29, Proposition 5.3.19]). We therefore may proceed as follows.

We start from the usual exact sequence

$$0 \longrightarrow M^\circ \longrightarrow M \longrightarrow \tilde{E}_M \longrightarrow 0,$$

where $\tilde{E}_M \subseteq E_M$. Since E_M does not contain any non-trivial finite Λ-submodules, the submodule $\tilde{E}_M$ has projective dimension at most 1. Therefore we can use [28, Chapter 3, Theorem 22] in order to deduce from the exact sequence that

$$\mathcal{F}_\Lambda(M) = \mathcal{F}_\Lambda(\tilde{E}_M) \cdot \mathcal{F}_\Lambda(M^\circ).$$

Moreover, since $\tilde{E}_M$ has projective dimension at most one, it follows from [13, Lemma A.7] that $\mathcal{F}_\Lambda(\tilde{E}_M)$ equals the characteristic ideal of $\tilde{E}_M$. As $\tilde{E}_M$ is pseudo-isomorphic to E_M and M, we may conclude that

$$\mathcal{F}_\Lambda(\tilde{E}_M) = (F_M).$$

This concludes the proof of the proposition. □

2.3. Selmer Groups and Fine Selmer Groups

Fix a number field K, a prime number p, and let A be an abelian variety defined over K. Also fix a finite set Σ of primes of K which contains the primes above p and the set $\Sigma_{\mathrm{br}}(A)$ of primes where A has bad reduction. If $p = 2$, then we assume that K is totally imaginary. For any number field L and a prime w of L, we denote by L_w the completion of L at w.

Let K_∞ be a $\mathbb{Z}_p^d$-extension of K, $d \geq 1$, with intermediate fields K_n (i.e., $\mathrm{Gal}(K_n/K)$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^d$ for every $n \in \mathbb{N}$). Then we define the *Selmer group* of A over K_n as

$$\mathrm{Sel}_A(K_n) := \ker\left(H^1(K_\Sigma/K_n, A[p^\infty]) \longrightarrow \prod_{v \in \Sigma(K_n)} H^1(K_{n,v}, A)[p^\infty]\right),$$

where K_Σ is defined as in Sect. 2.1 (note that $K_\infty \subseteq K_\Sigma$ because Σ contains the primes above p by assumption). This definition does not depend on the choice of Σ, as long as Σ contains $\Sigma_p(K)$ and $\Sigma_{\mathrm{ram}}(A)$ (see [25, Corollary 6.6]).

Moreover, we define the *fine Selmer group* of A over K_n as

$$\mathrm{Sel}_{A,0}(K_n) := \ker\left(H^1(K_\Sigma/K_n, A[p^\infty]) \longrightarrow \prod_{v \in \Sigma(K_n)} H^1(K_{n,v}, A[p^\infty])\right).$$

This definition does not depend on the set Σ, for it can be seen that for any finite set Σ containing the primes above p and the set $\Sigma_{\mathrm{br}}(A)$ we have an exact sequence

$$0 \longrightarrow \mathrm{Sel}_{A,0}(K_n) \longrightarrow \mathrm{Sel}_A(K_n) \longrightarrow \prod_{v|p} H^1(K_{n,v}, A[p^\infty]).$$

Now let

$$X_A^{(K_n)} = \mathrm{Sel}_A(K_n)^\vee$$

and

$$Y_A^{(K_n)} = \mathrm{Sel}_{A,0}(K_n)^\vee$$

be the Pontryagin duals. Finally, we define

$$X_A^{(K_\infty)} = \varprojlim_n X_A^{(K_n)}$$

and

$$Y_A^{(K_\infty)} = \varprojlim_n Y_A^{(K_n)},$$

where the projective limits are taken with respect to the dual of the restriction maps from cohomology. It is well-known that both $X_A^{(K_\infty)}$ and $Y_A^{(K_\infty)}$ are finitely generated Λ_d-modules (as in the previous subsection, we identify $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$ with Λ_d).

2.4. A Topology on the Set of $\mathbb{Z}_p$-Extensions of K

For any $\mathbb{Z}_p^d$-extension $\mathbb{L}_\infty$ of a number field K, we let $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ denote the set of $\mathbb{Z}_p$-extensions of K which are contained in $\mathbb{L}_\infty$. In the following we assume that $d \geq 2$, so that the set $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ is infinite. It follows from [8, p. 208] that this set is compact with respect to the following topology (which we will call *Greenberg's topology*). A basis of this topology is given by the following sets: For any $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ and every $n \in \mathbb{N}$, we define

$$\mathcal{E}(K_\infty, n) = \left\{ \tilde{K}_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K) \,\middle|\, [(\tilde{K}_\infty \cap K_\infty) : K] \geq p^n \right\}.$$

In other words, $\mathcal{E}(K_\infty, n)$ contains the $\mathbb{Z}_p$-extensions of K which are subextensions of $\mathbb{L}_\infty$ and which coincide with K_∞ at least up to the nth layer. Two $\mathbb{Z}_p$-extensions of K are close with respect to Greenberg's topology if they have a large intersection, i.e. if they share a large number of common layers.

For any $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$, the restriction map $\mathrm{Gal}(\mathbb{L}_\infty/K) \twoheadrightarrow \mathrm{Gal}(K_\infty/K)$ on the Galois groups induces a canonical surjection

$$\pi_{K_\infty} \colon \Lambda_d = \mathbb{Z}_p[[T_1, \ldots, T_d]] \twoheadrightarrow \Lambda_1 = \mathbb{Z}_p[[S]],$$

and the kernel of this map is the ideal of Λ_d generated by the elements

$$\sigma_1 - 1, \ldots, \sigma_{d-1} - 1,$$

where $\sigma_1, \ldots, \sigma_{d-1}$ are topological generators of the subgroup

$$\mathrm{Gal}(\mathbb{L}_\infty/K_\infty) \subseteq \mathrm{Gal}(\mathbb{L}_\infty/K)$$

fixing K_∞. For every finitely generated Λ_d-module M, the quotient

$$M_{K_\infty} := M/(\sigma_1 - 1, \ldots, \sigma_{d-1} - 1)M \tag{5}$$

is a finitely generated $\mathbb{Z}_p[[S]]$-module.

Now we focus on the case $d = 2$. Fix topological generators σ_1 and σ_2 of $G := \mathrm{Gal}(\mathbb{L}_\infty/K) \cong \mathbb{Z}_p^2$. For any $\mathbb{Z}_p$-extension $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$, let σ_{K_∞} be a topological generator of $\mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$. We consider the surjective map

$$\pi_{K_\infty} \colon \mathrm{Gal}(\mathbb{L}_\infty/K) \longrightarrow \mathrm{Gal}(K_\infty/K).$$

Then

$$\pi_{K_\infty}(\sigma_1) = \sigma_{K_\infty}^{a_1}, \quad \pi_{K_\infty}(\sigma_2) = \sigma_{K_\infty}^{a_2}$$

for suitable $a_1, a_2 \in \mathbb{Z}_p$, and not both of a_1 and a_2 are divisible by p (since π_{K_∞} is surjective). In fact, $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ may be identified with the class $[a_1 : a_2]$ of the pair (a_1, a_2) in a projective space, since the topological generator of $\mathrm{Gal}(K_\infty/K)$ is unique only up to raising to a power with a unit exponent. Note that the tuple $[a_1 : a_2]$ uniquely determines the kernel $\mathrm{Gal}(\mathbb{L}_\infty/K_\infty)$ of π_{K_∞}. Therefore the set $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ can be identified with the projective one-dimensional space $\mathbb{P}^1(\mathbb{Z}_p)$ over $\mathbb{Z}_p$.

More generally, if $\mathbb{L}_\infty/K$ is a $\mathbb{Z}_p^d$-extension with $d \geq 2$ arbitrary and $K_\infty \subseteq \mathbb{L}_\infty$ is a $\mathbb{Z}_p$-extension of K, then the images of a fixed set of topological generators $\sigma_1, \ldots, \sigma_d$ of $\mathrm{Gal}(\mathbb{L}_\infty/K) \cong \mathbb{Z}_p^d$ under the map π_{K_∞} are of the form

$$\sigma_{K_\infty}^{a_1}, \ldots, \sigma_{K_\infty}^{a_d},$$

where $a_1, \ldots, a_d \in \mathbb{Z}_p$ are not all divisible by p. As in the special case $d = 2$, we can identify K_∞ with the element $[a_1 : \cdots : a_d] \in \mathbb{P}^{d-1}(\mathbb{Z}_p)$.

Let $T_i = \sigma_i - 1$ for each $i \in \{1, \ldots, d\}$. Moreover, we will use the notation

$$T_{K_\infty} = \sigma_{K_\infty} - 1.$$

Then the restriction map

$$\pi_{K_\infty} \colon \Lambda_d \twoheadrightarrow \Lambda_1$$

maps T_i to $(T_{K_\infty} + 1)^{a_i} - 1$, respectively. In [26], Monsky usually denotes the image of an element $f \in \Lambda_d$ under π_{K_∞} by f_a, where $a = [a_1 : \cdots : a_d] \in \mathbb{P}^{d-1}(\mathbb{Z}_p)$ corresponds to the $\mathbb{Z}_p$-extension K_∞ of K, as above.

In particular, if the topological generators $\sigma_1, \ldots, \sigma_d$ of $\mathrm{Gal}(\mathbb{L}_\infty/K)$ have been chosen such that $\sigma_1, \ldots, \sigma_{d-1}$ generate the subgroup $\mathrm{Gal}(\mathbb{L}_\infty/K_\infty)$ fixing K_∞, then the kernel of π_{K_∞} is generated by $T_1, \ldots, T_{d-1}$, i.e. with this choice of topological generators of $\mathrm{Gal}(\mathbb{L}_\infty/K)$, the $\mathbb{Z}_p$-extension K_∞ of K corresponds to

$$a = [0 : \cdots : 0 : 1] \in \mathbb{P}^{d-1}(\mathbb{Z}_p).$$

Therefore

$$M_{K_\infty} = M/(\ker(\pi_{K_\infty}) \cdot M)$$

for every finitely generated Λ_d-module M, where M_{K_∞} is defined as in (5).

Remark 2.2. Choosing a different topological generator of $\mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_p$ does not change the Iwasawa invariants of an element of $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]] \cong \Lambda_1$. In particular, such a change of variables does not affect the μ- or λ-invariant of a quotient module M_{K_∞}.

Using the above notation, the $\mathbb{Z}_p$-extensions $\tilde{K}_\infty \in \mathcal{E}(K_\infty, n)$, $n \in \mathbb{N}$, correspond to surjections $\pi_{\tilde{K}_\infty}$ that map the topological generators of $\mathrm{Gal}(\mathbb{L}_\infty/K_\infty)$ to elements in $\mathrm{Gal}(\tilde{K}_\infty/K)^{p^n}$. Recall that in the above, $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ has been identified with $[a_1 : \cdots : a_d] \in \mathbb{P}^{d-1}(\mathbb{Z}_p)$. Then the $\mathbb{Z}_p$-extensions $\tilde{K}_\infty$ in the neighbourhood $\mathcal{E}(K_\infty, n)$ of K_∞ correspond to elements $[b_1 : \cdots : b_d] \in \mathbb{P}^{d-1}(\mathbb{Z}_p)$ such that

$$b_i \equiv a_i \pmod{p^n}$$

for every i.

3. Analogues of Monsky's Boundedness Results

In [26], Monsky proved that the μ-invariants of the $\mathbb{Z}_p$-extensions of a number field K are bounded, and he obtained sufficient criteria for the λ-invariants to be bounded as one runs over the $\mathbb{Z}_p$-extensions of K which are contained in some fixed $\mathbb{Z}_p^2$-extension of K.

In this section, we prove analogues of Monsky's results in the setting of fine Selmer groups and Selmer groups. The module-theoretic results from [26, Sections 2 and 3] carry over immediately; we therefore state these results without proof in the first subsection. One of these results is then generalised from the $d = 2$ case to arbitrary $d \geq 2$. The second subsection contains the control theorems which we need, and in the final subsection we focus on the remaining parts of Monsky's proof, which have to be adapted to the Selmer group setting.

3.1. Auxiliary Results on Iwasawa Modules Over $\mathbb{Z}_p^d$-Extensions

Fix a $\mathbb{Z}_p^d$-extension $\mathbb{L}_\infty/K$. As in Sect. 2.4, we write $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ for the set of $\mathbb{Z}_p$-extensions of K which are contained in $\mathbb{L}_\infty$. Let M be a finitely generated torsion Λ_d-module (as usual, we identify $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{L}_\infty/K)]]$ with Λ_d), and let F_M be the characteristic power series of M. We recall that for every $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$, the quotient $M_{K_\infty} = M/(\ker(\pi_{K_\infty}) \cdot M)$ (see Sect. 2.4) is a finitely generated, but not necessarily torsion, $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$-module.

Lemma 3.1. *There exist non-trivial elements $\sigma_1, \ldots, \sigma_l \in \mathrm{Gal}(\mathbb{L}_\infty/K)$ such that for every $\mathbb{Z}_p$-extension $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ of K such that $\sigma_i|_{K_\infty} \neq 1$ for each i, the quotient M_{K_∞} is a torsion Λ_1-module and satisfies*

$$\mu(M_{K_\infty}) = m_0(M).$$

Proof. This is [26, Theorem 3.2]. □

In the next lemma we restrict to the case $d = 2$.

Lemma 3.2. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^2$-extension, let $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ correspond to $a = [a_1 : a_2] \in \mathbb{P}^1(\mathbb{Z}_p)$ (as in Sect. 2.4), and let $\pi_{K_\infty} = \pi_a \colon \Lambda_2 \longrightarrow \Lambda_1$ be the canonical surjection induced by the restriction map on the Galois groups. Choose a topological generator σ of $\mathrm{Gal}(\mathbb{L}_\infty/K_\infty)$, and let $T = \sigma - 1$. We assume that $M_{K_\infty} = M/(\ker(\pi_{K_\infty}) \cdot M)$ is a torsion Λ-module, and we write $F_M = p^{m_0(M)} \cdot G_M$, where $p \nmid G_M$.*

Then

$$\lambda(M_{K_\infty}) = \lambda((G_M)_a)$$

is unbounded in any neighbourhood of K_∞ if and only if the image $\overline{G_M} \in \Lambda_2/p\Lambda_2$ of G_M is divisible by $\overline{T}$.

Proof. See [26, Theorem 3.3]. □

Monsky's Lemma 3.2 can be generalised to $\mathbb{Z}_p^d$-extensions, $d \geq 2$ arbitrary (see Lemma 3.5 below). In order to prove this generalisation, we first prove two auxiliary lemmas.

Lemma 3.3. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, $d \geq 2$, and identify $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{L}_\infty/K)]]$ with Λ_d.Let M be a finitely generated and torsion Λ_d-module.Let $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ correspond to $a = [a_1 : \cdots : a_d] \in \mathbb{P}^{d-1}(\mathbb{Z}_p)$, and let*

$$\pi_{K_\infty} = \pi_a \colon \Lambda_d \longrightarrow \Lambda$$

be the canonical surjective map induced by the restriction map on the Galois groups. If $M_{K_\infty} = M/(\ker(\pi_{K_\infty}) \cdot M)$ is a torsion Λ-module, then there exists a neighbourhood $\mathcal{E}(K_\infty, n)$ such that for all $K'_\infty \in \mathcal{E}(K_\infty, n)$, $M_{K'_\infty}$ is a torsion Λ-module.

Proof. It follows from Proposition 2.1(4) that the Fitting ideal of the quotient M_{K_∞} in $\Lambda \cong \Lambda_d/\ker(\pi_{K_\infty})$ is given by the image of $\mathcal{F}_{\Lambda_d}(M)$ under π_{K_∞}. By hypothesis, there exists an element $F \neq 0$ in the annihilator ideal of M_{K_∞}. If M_{K_∞} can be generated as a Λ-module by l elements, then $H' := F^l$ is contained in the Fitting ideal of M_{K_∞} by Proposition 2.1(2). Since

$$\pi_{K_\infty} \colon \mathcal{F}_{\Lambda_d}(M) \longrightarrow \mathcal{F}_\Lambda(M_{K_\infty})$$

is surjective, we can choose a pre-image $H \in \mathcal{F}_{\Lambda_d}(M)$ of H' under the map π_{K_∞}. In particular, $H' = \pi_{K_\infty}(H)$ is non-zero. By the Weierstrass Preparation Theorem, we may assume that $H' = p^x \cdot G$ for some distinguished polynomial G.

Let $m \in \mathbb{N}$ be large enough such that $m > x$ and $p^m > \deg(G)$, and choose $n = 2\,m$. Since

$$(T+1)^{p^n} - 1 \equiv 0 \mod (p^m, T^{p^m}),$$

we may conclude that

$$\pi_{\tilde{K}_\infty}(H) \equiv H' \mod (p^m, T^{p^m})$$

is non-zero for each $\tilde{K}_\infty \in \mathcal{E}(K_\infty, n)$ (we identify each quotient

$$\Lambda_d / \ker(\pi_{\tilde{K}_\infty}) \cong \mathbb{Z}_p[[\mathrm{Gal}(\tilde{K}_\infty/K)]]$$

with $\Lambda = \mathbb{Z}_p[[T]]$, and therefore this congruence makes sense as a statement in Λ). Since $\pi_{\tilde{K}_\infty}(H)$ is contained in $\mathcal{F}_\Lambda(M_{\tilde{K}_\infty}) \subseteq \mathrm{Ann}_\Lambda(M_{\tilde{K}_\infty})$, this proves the lemma. □

Lemma 3.4. *Let $G \in \Lambda_d$ be such that the coset $\overline{G}$ of G in the quotient algebra $\Omega_d = \Lambda_d/p\Lambda_d = \mathbb{F}_p[[T_1, \ldots, T_d]]$ is non-trivial (i.e., $p \nmid G$), and recall that $d \geq 2$. Then $\lambda(G_a)$ is unbounded in any neighbourhood of $[0 : \cdots : 0 : 1]$ if and only if $\overline{G} \in (T_1, \ldots, T_{d-1})$ (recall that $G_a = \pi_a(G)$).*

Proof. Let $K_\infty = \mathbb{L}_\infty^{\langle T_1+1, \ldots, T_{d-1}+1\rangle}$. Then

$$\overline{G} \in (T_1, \ldots, T_{d-1}) \quad \text{if and only if} \quad \pi_{K_\infty}(G) \equiv 0 \pmod p.$$

Suppose first that $\overline{G} \notin (T_1, \ldots, T_{d-1})$, and recall that

$$\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]] \cong \Lambda = \mathbb{Z}_p[[T]].$$

Then $\pi_a(G) = G(0, \ldots, 0, T)$ is not divisible by p, i.e. $\overline{G}(0, \ldots, 0, T)$ does not vanish identically. Let r be its T-order. It is then easy to see that $\lambda(G_a) \leq r$ on any sufficiently small neighbourhood of $[0 : \cdots : 0 : 1]$.

Now suppose that $\overline{G} \in (T_1, \ldots, T_{d-1})$ and let U be a neighbourhood of K_∞ as in Lemma 3.3, which we identify with a neighbourhood of $[0 : \cdots : 0 : 1]$. Choose $a = [p^{j_1} : \cdots : p^{j_{d-1}} : 1] \in U$, with j_i large for all i. Since

$$(T+1)^{p^j} - 1 \equiv T^{p^j} \pmod p$$

for every $j \in \mathbb{N}$, it follows that $\pi_a(T_i) \in (p, T^{p^{j_i}})$ for each $i \in \{1, \ldots, d-1\}$. Since G is a Λ_d-linear combination of p and $T_1, \ldots, T_{d-1}$ by assumption, it follows that

$$G_a = \pi_a(G) \in (p, T^{p^j}),$$

where $j = \min(j_1, \ldots, j_{d-1})$. In particular, if $\overline{G}_a \neq \overline{0}$ in Ω_d, then $\lambda(G_a) \geq j$.

It remains to prove that $\overline{G}_a \neq \overline{0}$ for $a = [p^{j_1} : \cdots : p^{j_{d-1}} : 1]$ and arbitrarily large j_i. We prove this claim via induction on $d \geq 2$. Let first $d = 2$, and let $m > 0$ be given (we look for some $j \geq m$ for which our claim holds true). We may write

$$\overline{G}(T_1, T_2) = T_1^t \cdot (F_1 + F_2),$$

where F_1 is divisible by T_1 and $F_2 \in \mathbb{F}_p[[T_2]]\backslash\{\overline{0}\}$ (i.e. we are factoring out T_1^t, the largest power of T_1 dividing $\overline{G}$ in the unique factorisation domain Ω_d).

If $F_1 = \overline{0}$, then clearly $\overline{G}_a \neq \overline{0}$ for any $a = (p^j, 1)$. Now assume that $F_1 \neq \overline{0}$, and let s be the largest power of T dividing $(F_2)_{[0:1]}$, so that

$$(F_2)_{[0:1]} = c \cdot T^s + \text{higher order terms},$$

with $c \neq 0$. Note that $(F_2)_{[0:1]} = F_2$ in the ring $\mathbb{F}_p[[T_2]]$. Let $j \geq m$ be arbitrary such that $p^j > s$, and let $a = [p^j : 1]$. Since T_1 divides F_1, the exponent of T in every term in $(F_1)_a \in \mathbb{F}_p[[T]]$ is larger than s. Moreover, since $F_2 \in \mathbb{F}_p[[T_2]] \subseteq \mathbb{F}_p[[T_1, T_2]]$ does not depend on the variable T_1, we have $(F_2)_a = (F_2)_{[0:1]}$. Therefore

$$(F_1)_a + (F_2)_a = (F_1)_a + (F_2)_{[0:1]} = cT^s + \text{higher order terms},$$

and thus $\overline{G}_a \neq \overline{0}$. This proves our claim for $d = 2$.

Now suppose that the claim holds for all $i < d$, and let $m > 0$ be given. As in the base step, we may write

$$\overline{G}(T_1, \ldots, T_d) = T_1^t \cdot (F_1 + F_2),$$

where F_1 is divisible by T_1 and $F_2 \in \mathbb{F}_p[[T_2, \ldots, T_d]]\backslash\{\overline{0}\}$. By the inductive step, we have $(F_2)_a \neq \overline{0}$ for $a = [p^{j_2} : \cdots : p^{j_{d-1}} : 1] \in \mathbb{P}^{d-2}(\mathbb{Z}_p)$ if all $j_i \geq m$ are sufficiently large.

If $F_1 = \overline{0}$, then we let $b = [p^{j_1} : \cdots : p^{j_{d-1}} : 1] \in \mathbb{P}^{d-1}(\mathbb{Z}_p)$ for any j_1. In this case we clearly have $\overline{G}_b \neq \overline{0}$, and we are done. Otherwise we have $F_1 \neq \overline{0}$. Let s be the largest power of T dividing $(F_2)_a$, so that

$$(F_2)_a = c \cdot T^s + \text{higher order terms},$$

with $c \neq 0$. Now choose $j_1 \geq \min(j_2, \ldots, j_{d-1})$ large enough such that $p^{j_1} > s$. Let $b = [p^{j_1} : \cdots : p^{j_{d-1}} : 1]$. Since T_1 divides F_1, the exponent of T in each term of $(F_1)_b$ is larger than s. Therefore

$$(F_1)_b + (F_2)_b = (F_1)_b + (F_2)_a = c \cdot T^s + \text{higher order terms}.$$

It follows that $\overline{G}_b \neq \overline{0}$, which completes the proof. □

Lemma 3.5. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, $d \geq 2$, and identify $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{L}_\infty/K)]]$ with Λ_d. Let F_M be the characteristic power series of the finitely generated torsion Λ_d-module M, and write $F_M = p^{m_0(M)} \cdot G_M$ with $p \nmid G_M$.*

Let $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ correspond to $a = [a_1 : \cdots : a_d] \in \mathbb{P}^{d-1}(\mathbb{Z}_p)$, and let

$$\pi_{K_\infty} = \pi_a \colon \Lambda_d \longrightarrow \Lambda$$

be the canonical surjective map induced by the restriction map on the Galois groups. We assume that $M_{K_\infty} = M/(\ker(\pi_{K_\infty}) \cdot M)$ is a torsion Λ-module.

(a) *If $\pi_{K_\infty}(G_M) \equiv 0 \pmod{p}$, then $\lambda(M_{K_\infty})$ is unbounded in any neighbourhood of K_∞.*

(b) *Suppose that there exists an element $H \in \Lambda_d$ such that $p^s \cdot H$ annihilates the maximal pseudo-null submodule M° of M for a sufficiently large $s \in \mathbb{N}$, and such that $\pi_{K_\infty}(H) \not\equiv 0 \pmod p$.*
Then the reverse conclusion also holds, i.e. if $\pi_{K_\infty}(G_M) \not\equiv 0 \pmod p$, then $\lambda(M_{K_\infty})$ is bounded in a sufficiently small neighbourhood of K_∞.

Remark 3.6. If $d = 2$, then the additional assumption on the maximal pseudo-null submodule in Lemma 3.5 is always satisfied (cf. also the proof of [26, Theorem 3.3]).

Remark 3.7. In Lemmas 3.2 and 3.5 there is a statement that $\lambda(M_{K_\infty})$ is unbounded (or bounded) in a neighbourhood of K_∞. Implicit in this statement is the fact that there is a neighbourhood of $\mathcal{E}(K_\infty, n)$ such that $M_{K'_\infty}$ is a torsion Λ-module for all $K'_\infty \in \mathcal{E}(K_\infty, n)$. This is true by Lemma 3.3.

Proof of lemma 3.5. Without loss of generality, we may choose the topological generators of $\mathrm{Gal}(\mathbb{L}_\infty/K) \cong \mathbb{Z}_p^d$ such that $a = [0 : \cdots : 0 : 1]$ (see also Sect. 2.4). As in [26], we work with the zero-th Fitting ideal $\mathcal{F}_{\Lambda_d}(M)$ of M. Recall from Proposition 2.1(4) that

$$\mathcal{F}_{\Lambda_1}(M_a) = \pi_a(\mathcal{F}_{\Lambda_d}(M)).$$

Suppose first that $\overline{G_M} \in (T_1, \ldots, T_{d-1})$. Assume that M_a is a torsion Λ-module. Since (G_M) divides $\mathcal{F}_{\Lambda_d}(M)$, the ideal $((G_M)_a)$ of Λ divides $\mathcal{F}_{\Lambda_d}(M)_a = \mathcal{F}_{\Lambda_1}(M_a)$. Therefore $\lambda((G_M)_a) \le \lambda(M_a)$. It follows from Lemma 3.4 that $\lambda((G_M)_a)$ and consequently also $\lambda(M_a)$ are unbounded in any neighbourhood U of $[0 : \cdots : 0 : 1]$ in which M_a is a torsion Λ-module for each $a \in U$. Such a neighbourhood exists by Lemma 3.3.

Now suppose that we are in the setting of assertion (b), and that $\lambda(M_a)$ is unbounded in any neighbourhood of $[0 : \cdots : 0 : 1]$. Suppose that M is generated by l elements as a Λ_d-module. Then Proposition 2.1(3) implies that

$$p^{l \cdot m_0(M) + ls} \cdot H^l \cdot G_M^l \in \mathcal{F}_{\Lambda_d}(M),$$

so $p^{lm_0(M)+ls} H_a^l (G_M)_a^l \in \mathcal{F}_{\Lambda_1}(M_a)$. It follows that $\lambda(M_a) \le \lambda(H_a^l \cdot (G_M)_a^l)$. Since $\overline{H}$ is not contained in the prime ideal $(T_1, \ldots, T_{d-1})$ of Ω_d, it follows from Lemma 3.4 that $\overline{G}_M \in (T_1, \ldots, T_{d-1})$. □

Remark 3.8. It follows from the proof of Lemma 3.5 that in the statement of Lemmas 3.5 and 3.2 *unbounded in any neighbourhood* may be replaced by *unbounded in some neighbourhood*, i.e. these statements are equivalent.

Remark 3.9. The additional hypothesis on the existence of H in assertion (b) is also necessary, as the following example shows which we take from [17].

Suppose that $d = 3$ and $M = \Lambda_3/(T_1, T_2 + p)$. Then M is a pseudo-null Λ_3-module, and therefore $F_M = G_M = 1$, i.e. $\overline{G}_M$ is *not* contained in (T_1, T_2). However, we have seen in [17, Example 6.3] that $\lambda(M_a)$ is unbounded in a neighbourhood of $[0 : 0 : 1]$. Each annihilator H of $M^\circ = M$ is contained in the ideal (p, T_1, T_2) of Λ_3.

It will be one of our tasks in the next section to derive natural hypotheses which are sufficient for the existence of the element H in Lemma 3.5 (of course, it will be sufficient if M does not contain any non-trivial pseudo-null submodules, i.e. $M^\circ = \{0\}$, but we try to do better, so also Remark 3.23 below).

3.2. Control Theorems

For the remainder of this section, we fix a number field K and an abelian variety A of dimension g defined over K. If $p = 2$, then we assume that K is totally imaginary. Let Σ be a finite set of primes of K which contains the primes above p and the primes where A has bad reduction.

Lemma 3.10. *Let $G \cong \mathbb{Z}_p^d$ and let M be a discrete G-module that is cofinitely generated over $\mathbb{Z}_p$. Let $m = \mathrm{corank}_{\mathbb{Z}_p}(M)$. Then $H^1(G, M)$ and $H^2(G, M)$ are cofinitely generated over $\mathbb{Z}_p$ with*

$$\mathrm{corank}_{\mathbb{Z}_p}(H^1(G, M)) \leq md$$

and

$$\mathrm{corank}_{\mathbb{Z}_p}(H^2(G, M)) \leq md(d-1)/2.$$

Proof. First we prove the result for $H^1(G, M)$ by induction on d. For $d = 1$ we have that $H^1(G, M) = M/(\sigma - 1)M$ where σ is a topological generator of G. Therefore $\mathrm{corank}_{\mathbb{Z}_p}(H^1(G, M)) \leq m$. Now assume that the result is true for $d-1$. Let H be a subgroup of G that is isomorphic to $\mathbb{Z}_p^{d-1}$ with $G/H \cong \mathbb{Z}_p$. The desired result then follows from the Hochschild-Serre spectral sequence $H^i(G/H, H^j(H, M)) \Rightarrow H^{i+j}(G, M)$.

Now we prove the result for $H^2(G, M)$ by induction on d. Since $cd_p(\mathbb{Z}_p) = 1$, the result is true for $d = 1$. Now assume that the result is true for $d - 1$. Let H be a subgroup of G that is isomorphic to $\mathbb{Z}_p^{d-1}$ with $G/H \cong \mathbb{Z}_p$. By the Hochschild-Serre spectral sequence $H^i(G/H, H^j(H, M)) \Rightarrow H^{i+j}(G, M)$, it will suffice to show that

$$\mathrm{corank}_{\mathbb{Z}_p}(H^2(G/H, H^0(H, M)) + \mathrm{corank}_{\mathbb{Z}_p}(H^1(G/H, H^1(H, M)) \\ + \mathrm{corank}_{\mathbb{Z}_p}(H^0(G/H, H^2(H, M)) \leq md(d-1)/2.$$

The first term is zero because $cd_p(G/H) = 1$. Using the result just proven for the first cohomology group we get that $\mathrm{corank}_{\mathbb{Z}_p}(H^1(G/H, H^1(H, M)) \leq m(d-1)$. By the induction hypothesis we get $\mathrm{corank}_{\mathbb{Z}_p}(H^0(G/H, H^2(H, M)) \leq m(d-1)(d-2)/2$. Therefore, as desired, the sum is at most $md(d-1)/2$. □

Now we can prove a control theorem for fine Selmer groups.

Lemma 3.11. *Let A be an abelian variety of dimension g defined over K. Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension ($d \geq 2$), $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ and write $Y = Y_A^{(\mathbb{L}_\infty)}$ for brevity. Consider the dual of the restriction map*

$$f^\vee : Y_{K_\infty} \longrightarrow Y_A^{(K_\infty)}.$$

We have

(a) $\mathrm{coker} f^\vee$ *is a finitely generated* $\mathbb{Z}_p$*-module with*

$$\mathrm{rank}_{\mathbb{Z}_p}(\mathrm{coker} f^\vee) \leq 2g(d-1)$$

(b) *Let* $m_p = \sum_{v|p} m_v$ *where the sum runs over the primes of* K *above* p *and* m_v *is defined to be zero if* v *splits completely in* K_∞ *and equal to the number of primes of* K_∞ *above* v *otherwise. Then* $\ker f^\vee$ *is a finitely generated torsion* Λ*-module with*

$$\lambda(\ker f^\vee) \leq 2\,g(d-1)m_p + g(d-1)(d-2).$$

Proof Let $\Gamma_\infty = \mathrm{Gal}(\mathbb{L}_\infty/K_\infty)$ and consider the following canonical commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Sel}_{A,0}(K_\infty) & \longrightarrow & H^1(K_\Sigma/K_\infty, A[p^\infty]) & \xrightarrow{\rho_0} & \bigoplus_{w\in\Sigma(K_\infty)} H^1(K_{\infty,w}, A[p^\infty]) \\ & \downarrow f & & \downarrow f' & & \downarrow f'' \\ 0 \longrightarrow & \mathrm{Sel}_{A,0}(\mathbb{L}_\infty)^{\Gamma_\infty} & \longrightarrow & H^1(\mathbb{L}_\Sigma/\mathbb{L}_\infty, A[p^\infty])^{\Gamma_\infty} & \longrightarrow & \left(\bigoplus_{w\in\Sigma(\mathbb{L}_\infty)} H^1(\mathbb{L}_{\infty,w}, A[p^\infty])\right)^{\Gamma_\infty} \end{array}$$

By the snake lemma we have an exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker f' \longrightarrow \ker f'' \cap \mathrm{img}\,\rho_0 \longrightarrow \mathrm{coker} f \longrightarrow \mathrm{coker} f'. \qquad (6)$$

According to the inflation-restriction exact sequence and Lemma 3.10 we have that

$$\mathrm{corank}_{\mathbb{Z}_p}(\ker f') = \mathrm{corank}_{\mathbb{Z}_p}(H^1(\Gamma_\infty, A(\mathbb{L}_\infty)[p^\infty])) \leq 2g(d-1).$$

Therefore from the exact sequence (6),

$$\mathrm{corank}_{\mathbb{Z}_p}(\ker f) \leq \mathrm{corank}_{\mathbb{Z}_p}(\ker f') \leq 2g(d-1).$$

This proves (a).

By the inflation-restriction sequence and Shapiro's lemma it follows that we can write $\ker f'' = \oplus_{v\in\Sigma(K)} B_v$ where $B_v = \oplus_{w|v} H^1(\Gamma_{\infty,w}, A(\mathbb{L}_{\infty,w})[p^\infty])$. In the definition of B_v the sum runs over all primes w of K_∞ above v. For each such w we have also written w for a fixed prime of $\mathbb{L}_\infty$ so that $\Gamma_{\infty,w}$ denotes the corresponding decomposition group.

Assume that $v \in \Sigma(K)$ splits completely in K_∞/K. Then for any prime w of $\mathbb{L}_\infty$ above v we have that $H^0(\Gamma_{\infty,w}, A(\mathbb{L}_{\infty,w})[p^\infty]) = A(K_v)[p^\infty]$ is finite. Therefore by [19, Lemma 6.2] we have that $H^1(\Gamma_{\infty,w}, A(\mathbb{L}_{\infty,w})[p^\infty])$ is finite and this order does not depend on the prime w of $\mathbb{L}_\infty$ above v. It follows that p^n annihilates B_v for some n.

Now assume that $v \in \Sigma(K)$ does not split completely in K_∞/K. If v does not lie above p, then any prime w of K_∞ above v splits completely in $\mathbb{L}_\infty/K_\infty$ so in this case $B_v = 0$. Now suppose that $v \in \Sigma_p(K)$ and let w be a prime of $\mathbb{L}_\infty$ above v. According to Lemma 3.10 $H^1(\Gamma_{\infty,w}, A(\mathbb{L}_{\infty,w})[p^\infty])$ is cofinitely generated over $\mathbb{Z}_p$ with corank at most $2\,g(d-1)$. It follows that B_v is cofinitely generated over $\mathbb{Z}_p$ with $\mathrm{corank}_{\mathbb{Z}_p}(B_v) \leq 2g(d-1)m_v$. Putting all of this together, we get that the Pontryagin dual of $\ker f''$ is Λ-torsion with λ-invariant at most $2g(d-1)m_p$.

By the inflation-restriction sequence $\mathrm{coker} f'$ injects into $H^2(\Gamma_\infty, A[p^\infty])$. So by Lemma 3.10 we have

$$\mathrm{corank}_{\mathbb{Z}_p}(\mathrm{coker} f') \leq \mathrm{corank}_{\mathbb{Z}_p}(H^2(\Gamma_\infty, A[p^\infty])) \leq 2g(d-1)(d-2)/2.$$

It therefore follows from the exact sequence (6) and the above observations that $\ker f^\vee$ is a finitely generated torsion Λ-module with

$$\lambda(\ker f^\vee) \leq 2g(d-1)m_p + 2g(d-1)(d-2)/2.$$

This completes the proof. □

Corollary 3.12. *Let K_∞ be as in Lemma 3.11, then we have*

(a) $\mathrm{rank}_\Lambda(Y_{K_\infty}) = \mathrm{rank}_\Lambda(Y_A^{(K_\infty)})$

(b) *Let $\Sigma_s(K)$ be the set of all primes $v \in \Sigma(K)$ that split completely in $\mathbb{L}_\infty/K$. If $\mathrm{rank}_\Lambda(Y_{K_\infty}) = \mathrm{rank}_\Lambda(Y_A^{(K_\infty)}) = 0$, then*

  (i) *$\mu(Y_{K_\infty}) \geq \mu(Y_A^{(K_\infty)})$ with equality if no prime $v \in \Sigma(K) \setminus \Sigma_s(K)$ splits completely in K_∞/K.*

  (ii) *If no prime $v \in \Sigma_p(K)$ splits completely in K_∞/K, then there exists a neighbourhood $\mathcal{E}(K_\infty, n)$ such that*

  - *for all $K'_\infty \in \mathcal{E}(K_\infty, n)$ we have*

$$\mathrm{rank}_\Lambda(Y_{K'_\infty}) = \mathrm{rank}_\Lambda(Y_A^{(K'_\infty)}) = 0,$$

  - *$|\lambda(Y_{K'_\infty}) - \lambda(Y_A^{(K'_\infty)})|$ is bounded as K'_∞ runs over $\mathcal{E}(K_\infty, n)$.*

Proof. Statement (a) follows directly from Lemma 3.11. Now assume that both Y_{K_∞} and $Y_A^{(K_\infty)}$ are Λ-torsion. From Lemma 3.11 we see that $\mu(Y_{K_\infty}) \geq \mu(Y_A^{(K_\infty)})$. The proof reveals that $\mu(\ker f^\vee) = 0$ if no prime $v \in \Sigma(K) \setminus \Sigma_s(K)$ splits completely in K_∞/K. This proves (b)-i.

Now assume that no prime $v \in \Sigma_p(K)$ splits completely in K_∞/K. Then we can choose a neighbourhood $\mathcal{E}(K_\infty, n)$ such that for any $K'_\infty \in \mathcal{E}(K_\infty, n)$ we have $\#\Sigma_p(K'_\infty) = \#\Sigma_p(K_\infty)$. By Lemma 3.3 we can reduce our neighbourhood if necessary so that for all $K'_\infty \in \mathcal{E}(K_\infty, n)$ we have $\mathrm{rank}_\Lambda(Y_A^{(K'_\infty)}) = 0$. Then (b)-ii follows from Lemma 3.11. □

In fact, the cokernels of $f^\vee$ can be bounded in a more general setting (this will prove useful below).

Corollary 3.13. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, and let $K_\infty \subseteq \mathbb{L}_\infty$ be a $\mathbb{Z}_p^i$-extension of K, $i \geq 1$. We write $Y = Y_A^{(\mathbb{L}_\infty)}$ and $H = \mathrm{Gal}(\mathbb{L}_\infty/K_\infty)$ for brevity. Then the cokernel of the natural map*

$$f^\vee : Y_H \longrightarrow Y_A^{(K_\infty)}$$

is a finitely generated $\mathbb{Z}_p$-module of rank at most $2g(d-i)$.

Proof. The proof of the first part of Lemma 3.11, using Lemma 3.10, goes through without changes. □

Now we turn to a control theorem for Selmer groups.

Lemma 3.14. *Let A be an abelian variety of dimension g defined over K. Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension ($d \geq 2$), $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ and write $X = X_A^{(\mathbb{L}_\infty)}$. Assume that A has good ordinary reduction at the primes of K above p, and that each $v \in \Sigma_p(K)$ is ramified in $\mathbb{L}_\infty/K$. We define r to be zero if each $v \in \Sigma_p(K)$ is ramified in K_∞/K and equal to one otherwise. Consider the dual of the restriction map*

$$f^\vee \colon X_{K_\infty} \longrightarrow X_A^{(K_\infty)}.$$

We have

(a) $\operatorname{coker} f^\vee$ *is a finitely generated $\mathbb{Z}_p$-module with*

$$\operatorname{rank}_{\mathbb{Z}_p}(\operatorname{coker} f^\vee) \leq 2gr(d-1).$$

(b) *Let $m_p = \sum_{v|p} m_v$ where the sum runs over the primes of K above p and m_v is defined to be zero if either v splits completely or ramifies in K_∞/K, and equal to the number primes of K_∞ above v otherwise. Then* $\ker f^\vee$ *is a finitely generated torsion Λ-module with*

$$\lambda(\ker f^\vee) \leq gdm_p + gr(d-1)(d-2).$$

Proof. We proceed as in the proof of Lemma 3.11. Let $\Gamma_\infty = \mathrm{Gal}(\mathbb{L}_\infty/K_\infty)$ and consider the following canonical commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Sel}_A(K_\infty) & \longrightarrow & H^1(K_\Sigma/K_\infty, A[p^\infty]) & \xrightarrow{\rho_0} & \bigoplus_{v\in\Sigma(K_\infty)} H^1(K_{\infty,v}, A)[p^\infty] \\
& \downarrow f & & \downarrow f' & & \downarrow f'' \\
0 \longrightarrow & \mathrm{Sel}_A(\mathbb{L}_\infty)^{\Gamma_\infty} & \longrightarrow & H^1(\mathbb{L}_\Sigma/\mathbb{L}_\infty, A[p^\infty])^{\Gamma_\infty} & \longrightarrow & (\bigoplus_{v\in\Sigma(\mathbb{L}_\infty)} H^1(\mathbb{L}_{\infty,w}, A)[p^\infty])^{\Gamma_\infty}
\end{array}$$

By the snake lemma we have an exact sequence

$$0 \longrightarrow \ker f \longrightarrow \ker f' \longrightarrow \ker f'' \cap \operatorname{img} \rho_0 \longrightarrow \operatorname{coker} f \longrightarrow \operatorname{coker} f'. \qquad (7)$$

If every prime $v \in \Sigma_p(K)$ ramifies in K_∞/K, then by [9, Prop. 3.2(ii)] we have that $A(K_\infty)[p^\infty]$ is finite. Therefore by [19, Lemma 6.2] $H^1(\Gamma_\infty, A(\mathbb{L}_\infty)[p^\infty])$ is finite. From this observation and Lemma 3.10 we conclude that

$$\operatorname{corank}_{\mathbb{Z}_p}(\ker f') = \operatorname{corank}_{\mathbb{Z}_p}(H^1(\Gamma_\infty, A(\mathbb{L}_\infty)[p^\infty])) \leq 2gr(d-1).$$

Therefore from the exact sequence (7),

$$\operatorname{corank}_{\mathbb{Z}_p}(\ker f) \leq \operatorname{corank}_{\mathbb{Z}_p}(\ker f') \leq 2gr(d-1).$$

This proves (a).

Now we deal with (b). We can write $\ker f'' = \oplus_{v\in\Sigma(K)} B_v$ where $B_v = \oplus_{w|v} \ker f''_w$ and $f''_w : H^1(K_{\infty,w}, A)[p^\infty] \longrightarrow H^1(\mathbb{L}_{\infty,w}, A)[p^\infty]$ is the restriction map. In the definition of B_v the sum runs over all primes w of K_∞ above v. For each such w we have also written w for a fixed prime of $\mathbb{L}_\infty$.

First, assume that $v \in \Sigma(K)$ does not lie above p. Let w be a prime of K_∞ above v and fix a prime of $\mathbb{L}_\infty$ above it which we also denote by w. Since $A(K_{\infty,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$ and $A(\mathbb{L}_{\infty,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$ we have isomorphisms

$$H^1(K_{\infty,w}, A)[p^\infty] \cong H^1(K_{\infty,w}, A[p^\infty]),$$
$$H^1(\mathbb{L}_{\infty,w}, A)[p^\infty] \cong H^1(\mathbb{L}_{\infty,w}, A[p^\infty]).$$

It follows that $\ker f''_w \cong H^1(\Gamma_{\infty,w}, A(\mathbb{L}_{\infty,w})[p^\infty])$. Therefore the same observations as those in the proof of Lemma 3.11 apply to B_v.

Now let $v \in \Sigma_p(K)$ and let w be a prime of K_∞ above v and fix a prime of $\mathbb{L}_\infty$ above it which we also denote by w. Let

$$\kappa_{K_\infty,w} : A(K_{\infty,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1(K_{\infty,w}, A[p^\infty]),$$
$$\kappa_{\mathbb{L}_\infty,w} : A(\mathbb{L}_{\infty,w}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1(\mathbb{L}_{\infty,w}, A[p^\infty])$$

be the Kummer maps. Then the map f''_w is

$$f''_w : H^1(K_{\infty,w}, A[p^\infty])/\operatorname{img}\kappa_{K_\infty,w} \longrightarrow H^1(\mathbb{L}_{\infty,w}, A[p^\infty])/\operatorname{img}\kappa_{\mathbb{L}_\infty,w}$$

Now let $C_w = \mathcal{F}(\bar{\mathfrak{m}})[p^\infty]$ where $\bar{\mathfrak{m}}$ is the maximal ideal of $\bar{K}_v$ and $\mathcal{F}$ is the formal group over O_{K_v} attached to the Néron model of A over O_{K_v}. The inclusion $C_w \subseteq A[p^\infty]$ induces maps

$$\lambda_{K_\infty,w} : H^1(K_{\infty,w}, C_w) \longrightarrow H^1(K_{\infty,w}, A[p^\infty]),$$
$$\lambda_{\mathbb{L}_\infty,w} : H^1(\mathbb{L}_{\infty,w}, C_w) \longrightarrow H^1(\mathbb{L}_{\infty,w}, A[p^\infty]).$$

Since v ramifies in $\mathbb{L}_\infty/K$, the extension $\mathbb{L}_{\infty,w}/K_v$ is deeply ramified in the sense of [4]. Therefore by [4, Proposition 4.3] and the discussion proceeding it we have $\operatorname{img}\kappa_{\mathbb{L}_\infty,w} = \operatorname{img}\lambda_{\mathbb{L}_\infty,w}$ and $\operatorname{img}\kappa_{K_\infty,w} \subseteq \operatorname{img}\lambda_{K_\infty,w}$. Therefore f''_w can be viewed as the composition of the following maps:

$$a_w : H^1(K_{\infty,w}, A[p^\infty])/\operatorname{img}\kappa_{K_\infty,w} \longrightarrow H^1(K_{\infty,w}, A[p^\infty])/\operatorname{img}\lambda_{K_\infty,w},$$
$$b_w : H^1(K_{\infty,w}, A[p^\infty])/\operatorname{img}\lambda_{K_\infty,w} \longrightarrow H^1(\mathbb{L}_{\infty,w}, A[p^\infty])/\operatorname{img}\lambda_{\mathbb{L}_\infty,w}.$$

We will now determine $\operatorname{corank}_{\mathbb{Z}_p}(\ker a_w)$ and $\operatorname{corank}_{\mathbb{Z}_p}(\ker b_w)$.

First we deal with $\ker b_w$. Let $\tilde{A}$ be the reduction of A over the residue field of an algebraic closure $\bar{K}_w$ of K_w. The exact sequence

$$0 \longrightarrow C_w \longrightarrow A[p^\infty] \longrightarrow \tilde{A}[p^\infty] \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow \operatorname{img}\lambda_{K_\infty,w} \longrightarrow H^1(K_{\infty,w}, A[p^\infty]) \longrightarrow H^1(K_{\infty,w}, \tilde{A}[p^\infty])$$

Similarly, we have an exact sequence

$$0 \longrightarrow \operatorname{img}\lambda_{\mathbb{L}_\infty,w} \longrightarrow H^1(\mathbb{L}_{\infty,w}, A[p^\infty]) \longrightarrow H^1(\mathbb{L}_{\infty,w}, \tilde{A}[p^\infty])$$

It follows that $\ker b_w$ is a subgroup of $H^1(\operatorname{Gal}(\mathbb{L}_{\infty,w}/K_{\infty,w}), \tilde{A}(l_{\infty,w})[p^\infty])$ where $l_{\infty,w}$ is the residue field of $\mathbb{L}_{\infty,w}$. If either v splits completely or ramifies in K_∞/K, then $H^0(\mathrm{Gal}(\mathbb{L}_{\infty,w}/K_{\infty,w}), \tilde{A}(l_{\infty_w})[p^\infty]) = \tilde{A}(k_{\infty,w})[p^\infty]$ is finite (here $k_{\infty,w}$ is the residue field of $K_{\infty,w}$). Therefore by [19, Lemma 6.2]

$$H^1(\mathrm{Gal}(\mathbb{L}_{\infty,w}/K_{\infty,w}), \tilde{A}(l_{\infty,w})[p^\infty])$$

is finite, whence $\ker b_w$ is finite. On the other hand when w does not split completely in K_∞/K we have by Lemma 3.10

$$\mathrm{corank}_{\mathbb{Z}_p}(\ker b_w) \le \mathrm{corank}_{\mathbb{Z}_p}(H^1(\mathrm{Gal}(\mathbb{L}_{\infty,w}/K_{\infty,w}), \tilde{A}(l_{\infty,w})[p^\infty])) \le g(d-1).$$

Now we deal with $\ker a_w = \mathrm{img}\,\lambda_{K_{\infty,w}}/\,\mathrm{img}\,\kappa_{K_{\infty,w}}$. Let L be finite extension of K_v contained in $K_{\infty,w}$. First we note that Tate local duality [29, Theorem 7.2.6] together with the Weil pairing yields a non-degenerate pairing

$$\langle\ ,\ \rangle : H^2(L, T_p(C_w)) \times H^0(L, \tilde{A}^t[p^\infty]) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

where $T_p(C_w)$ is the p-adic Tate module of C_w and A^t is the dual abelian variety. If L'/L is a finite extension, let $\mathrm{res} : H^2(L, T_p(C_w)) \longrightarrow H^2(L', T_p(C_w))$ be the restriction map and $\mathrm{cor} : H^0(L', \tilde{A}^t[p^\infty]) \longrightarrow H^0(L, \tilde{A}^t[p^\infty])$ be the corestriction (norm) map. For $a \in H^2(L, T_p(C_w))$ and $b \in H^0(L', \tilde{A}^t[p^\infty])$ a property of Tate local duality gives $\langle \mathrm{res}\, a, b\rangle = \langle a, \mathrm{cor}\, b\rangle$. As above, we have maps

$$\kappa_L : A(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1(L, A[p^\infty]),$$
$$\lambda_L : H^1(L, C_w) \longrightarrow H^1(L, A[p^\infty]).$$

Recall that for any Hausdorff abelian locally compact topological group M, we denote by $M^\vee$ its Pontryagin dual. Taking into account [4, Proposition 4.5], the proof of [4, Proposition 4.6] shows that we have an isomorphism

$$\theta_L : \mathrm{img}\,\lambda_L/\,\mathrm{img}\,\kappa_L \cong \tilde{A}^t(k_L)[p^\infty]^\vee,$$

where k_L is the residue field of L. Taking into account the property of Tate local duality above and the description of the map θ_L we have an isomorphism

$$\mathrm{img}\,\lambda_{K_{\infty,w}}/\,\mathrm{img}\,\kappa_{K_{\infty,w}} \cong \varinjlim \mathrm{img}\,\lambda_L/\,\mathrm{img}\,\kappa_L \cong (\varprojlim \tilde{A}^t(k_L)[p^\infty])^\vee.$$

The limits are taken over all finite extensions L/K_v inside $K_{\infty,w}/K_v$; the direct limits are taken with respect to restriction and inverse limits are taken with respect to corestriction.

If either v splits completely or ramifies in K_∞/K, then $\tilde{A}(k_{\infty,w})[p^\infty]$ is finite. Then from the above $\mathrm{img}\,\lambda_{K_{\infty,w}}/\,\mathrm{img}\,\kappa_{K_{\infty,w}} \cong (\varprojlim \tilde{A}^t(k_L)[p^\infty])^\vee$ is finite, so $\ker a_w$ is finite in this case. Since for any L as above we have $\tilde{A}^t(k_L)[p] \cong (\mathbb{Z}/p\mathbb{Z})^i$ with $i \le g$, therefore in the general case we have

$$\mathrm{corank}_{\mathbb{Z}_p}(\ker a_w) = \mathrm{corank}_{\mathbb{Z}_p}((\varprojlim \tilde{A}^t(k_L)[p^\infty])^\vee) \le g.$$

We have $f''_w = b_w \circ a_w$, so we have an exact sequence

$$0 \longrightarrow \ker a_w \longrightarrow \ker f''_w \longrightarrow \ker b_w.$$

Therefore $\mathrm{corank}_{\mathbb{Z}_p}(\ker f''_w) \le \mathrm{corank}_{\mathbb{Z}_p}(\ker a_w) + \mathrm{corank}_{\mathbb{Z}_p}(\ker b_w)$. From this and the above observations we see that if v either splits completely or ramifies in K_∞/K, then $\ker f''_w$ is finite. The order of this group does not depend on the prime w of $\mathbb{L}_\infty$ above v. It follows that in this case B_v is annihilated by some power of p. Otherwise when v does not split completely or ramify in K_∞/K the above observations show that we have $\mathrm{corank}_{\mathbb{Z}_p}(B_v) \le g(d-1)m_v + gm_v = gdm_v$.

By the inflation-restriction sequence $\mathrm{coker} f'$ injects into $H^2(\Gamma_\infty, A[p^\infty])$. So as in the proof of (a) using [9, Prop. 3.2(ii)] and Lemma 3.10 we have

$$\mathrm{corank}_{\mathbb{Z}_p}(\mathrm{coker} f') \le \mathrm{corank}_{\mathbb{Z}_p}(H^2(\Gamma_\infty, A[p^\infty])) \le 2gr(d-1)(d-2)/2.$$

It therefore follows from the exact sequence (7) and the above observations that $\ker f^\vee$ is a finitely generated torsion Λ-module with

$$\lambda(\ker f^\vee) \le gdm_p + 2gr(d-1)(d-2)/2.$$

This completes the proof. □

Corollary 3.15. *With the same setup and conditions as in Lemma 3.14, we have*

(a) $\mathrm{rank}_\Lambda(X_{K_\infty}) = \mathrm{rank}_\Lambda(X_A^{(K_\infty)})$.
(b) *Let $\Sigma_s(K)$ be the set of all primes $v \in \Sigma(K)$ that split completely in $\mathbb{L}_\infty/K$. If $\mathrm{rank}_\Lambda(X_{K_\infty}) = \mathrm{rank}_\Lambda(X_A^{(K_\infty)}) = 0$, then*
  (i) *$\mu(X_{K_\infty}) \ge \mu(X_A^{(K_\infty)})$ with equality if no prime $v \in \Sigma(K) \setminus \Sigma_s(K)$ splits completely in K_∞/K.*
  (ii) *If no prime $v \in \Sigma_p(K)$ splits completely in K_∞/K, then there exists a neighbourhood $\mathcal{E}(K_\infty, n)$ such that*
    - *for all $K'_\infty \in \mathcal{E}(K_\infty, n)$ we have*

$$\mathrm{rank}_\Lambda(X_{K'_\infty}) = \mathrm{rank}_\Lambda(X_A^{(K'_\infty)}) = 0$$

    - *$|\lambda(X_{K'_\infty}) - \lambda(X_A^{(K'_\infty)})|$ is bounded as K'_∞ runs over $\mathcal{E}(K_\infty, n)$.*

Proof. The proof is identical to that of Corollary 3.12 using Lemma 3.14. □

Again, we can prove a generalisation of the result for the coranks of $f^\vee$.

Corollary 3.16. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, and let $K_\infty \subseteq \mathbb{L}_\infty$ be a $\mathbb{Z}_p^i$-extension of K, for some $i \ge 1$. We write $X = X_A^{(\mathbb{L}_\infty)}$ and $H = \mathrm{Gal}(\mathbb{L}_\infty/K_\infty)$ for brevity. Suppose that A has good ordinary reduction at the primes of K above p, and that each $v \in \Sigma_p(K)$ is ramified in $\mathbb{L}_\infty$. We define r to be zero if the inertia subgroup of each $v \in \Sigma_p(K)$ is open in $\mathrm{Gal}(K_\infty/K)$, and equal to one otherwise.*

Then the cokernel of the natural map

$$f^\vee : X_H \longrightarrow X_A^{(K_\infty)}$$

is finitely generated over $\mathbb{Z}_p$ and of rank at most $2gr(d-i)$.

Proof. We proceed as in the proof of Lemma 3.14(a). Note that in the case $r = 0$ the result [9, Prop. 3.2(ii)] can be applied; in the case $r = 1$ we use Lemma 3.10 instead. □

For later use, we also state the following

Lemma 3.17. *Let Z denote either X or Y. If $Z = X$, we assume that A has good ordinary reduction at the prime of K above p, and that each $v \in \Sigma_p(K)$ is ramified in $\mathbb{L}_\infty/K$. Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, and suppose that there exists a $\mathbb{Z}_p$-extension $K_\infty \subseteq \mathbb{L}_\infty$ of K such that $Z_A^{(K_\infty)}$ is Λ-torsion. Then $Z_A^{(\mathbb{L}_\infty)}$ is a Λ_d-torsion module.*

Proof. We let $Z = Z_A^{(\mathbb{L}_\infty)}$. By [22, Lemma 4.7], it will suffice to show that Z_{K_∞} is Λ-torsion. The result now follows from Lemmas 3.11 or 3.14. □

3.3. The Main Results

Now we turn to the proofs of analogues of Monsky's Theorems I, II, III and IV from [26]. Since the results hold for Selmer groups as well as for fine Selmer groups, we introduce the following notational convention. In the following, Z will stand both for X and for Y (this enables us to formulate the results for Selmer groups and for fine Selmer groups simultaneously).

Lemma 3.18. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, $d \geq 2$. We assume that $Z_A^{(\mathbb{L}_\infty)}$ is a torsion Λ_d-module. If $Z = X$, we assume that A has good ordinary reduction at the primes of K above p, and that each $v \in \Sigma_p(K)$ is ramified in $\mathbb{L}_\infty/K$.*

Then $Z_A^{(K_\infty)}$ is a Λ-torsion module and $\mu(Z_A^{(K_\infty)}) \leq m_0(Z_A^{(\mathbb{L}_\infty)})$ for all elements $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ which are not contained in a finite number of $\mathbb{Z}_p^{d-1}$-subextensions of $\mathbb{L}_\infty/K$.

Proof. Choose $\sigma_1, \ldots, \sigma_l$ as in Lemma 3.1 (applied to $M = Y_A^{(\mathbb{L}_\infty)}$ or $M = X_A^{(\mathbb{L}_\infty)}$), let $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$, and let

$$\gamma \colon \mathrm{Gal}(\mathbb{L}_\infty/K) \twoheadrightarrow \mathrm{Gal}(K_\infty/K)$$

be the corresponding restriction homomorphism. Then $\gamma(\sigma_i) = 1$ if and only if K_∞ is contained in the fixed field of σ_i. Therefore the condition $\gamma(\sigma_i) \neq 1$ for all i holds for all $\mathbb{Z}_p$-extensions of K which are not contained in one of the $\mathbb{Z}_p^{d-1}$-extensions $\mathbb{L}_\infty^{\langle \sigma_1 \rangle}, \ldots, \mathbb{L}_\infty^{\langle \sigma_l \rangle}$ of K. The statement of the lemma follows by combining this observation with Corollaries 3.12 and 3.15. □

Remark 3.19. Let Σ be any finite set of primes of K which contains the primes above p and the primes where A has bad reduction. If $p = 2$, then we assume K to be totally imaginary. It follows from [22, Lemma 7.1] that $Y_A^{(\mathbb{L}_\infty)}$ is a torsion Λ_d-module if and only if $H^2(K_\Sigma/\mathbb{L}_\infty, A[p^\infty]) = 0$ (here we recall from Sect. 2.1 that K_Σ denotes the maximal algebraic extension of K which is unramified outside of Σ). The validity of either statement is known as the *weak Leopoldt conjecture* for A over $\mathbb{L}_\infty$. No example is known where this conjecture fails.

On the other hand, by a conjecture of Mazur (see [24, p. 104]), $X_A^{(K_\infty^c)}$ should be a torsion Λ-module if A has good ordinary reduction at the primes above p (here K_∞^c denotes the cyclotomic $\mathbb{Z}_p$-extension of K). This is known to be true for abelian K, by the work of Kato and Rohrlich (see [12,30]). It can be deduced from Lemma 3.17 that in this case, $X_A^{(\mathbb{L}_\infty)}$ is Λ_d-torsion for each $\mathbb{Z}_p^d$-extension $\mathbb{L}_\infty$ of K which contains K_∞^c.

Theorem 3.20. *Let $\mathbb{L}_\infty$ be a $\mathbb{Z}_p^d$-extension of K, $d \geq 2$. We assume that $Z_A^{(K_\infty)}$ is Λ-torsion for all but finitely many $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$. If $Z = X$, we assume that A has good ordinary reduction at the primes of K above p, and that the inertia subgroup of each $v \in \Sigma_p(K)$ has $\mathbb{Z}_p$-rank at least $d-1$.*

Then $\mu(Z_A^{(K_\infty)})$ is bounded on $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$.

Proof. The assertion is proved via induction. For the fine Selmer groups, it follows from Lemma 3.17 that $Y_A^{(\mathbb{L}_\infty)}$ is a torsion Λ_d-module. Therefore Lemma 3.18 implies that the $\mu(Y_A^{(K_\infty)})$ are bounded (by $m_0(Y_A^{(\mathbb{L}_\infty)})$) for all $\mathbb{Z}_p$-extensions of K which are not contained in a finite number of $\mathbb{Z}_p^{d-1}$-extensions of K. Again, Lemma 3.17 implies that for each of these $\mathbb{Z}_p^{d-1}$-extensions L_∞/K, the Iwasawa module $Y_A^{(L_\infty)}$ is torsion over Λ_{d-1}. By the inductive hypothesis, the μ-invariants of the $Y_A^{(K_\infty)}$ are bounded as K_∞ runs over the $\mathbb{Z}_p$-extensions contained in any of these $\mathbb{Z}_p^{d-1}$-extensions L_∞/K.

For the Selmer groups, the same proof goes through (note that in any given $\mathbb{Z}_p^2$-extension of K inside of $\mathbb{L}_\infty$, every prime $v \in \Sigma_p$ will ramify in view of our condition on the inertia subgroups. This is needed for the first inductive step). □

Now we turn to the study of l_0-invariants. Since we do not want to restrict to the case $d = 2$ (as in Monsky's paper), we consider the invariants $\widehat{l_0}$ which have been introduced in Sect. 2.2. Recall that for any finitely generated Λ_d-module M, $M^\circ \subseteq M$ denotes the maximal pseudo-null submodule. We also recall that each statement that the λ-invariant is bounded or unbounded in a neighbourhood $U \subseteq \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ always involves that $Z_{\tilde{K}_\infty}$ is Λ-torsion for each $\tilde{K}_\infty \in U$, see also Remark 3.7.

Lemma 3.21. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension for some $d \geq 2$, let $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ be such that no prime $v \in \Sigma_p(K)$ splits completely in K_∞/K, and let*

$$\pi_{K_\infty} \colon \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{L}_\infty/K)]] \longrightarrow \mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$$

be the canonical restriction map. We assume that $Z_A^{(\mathbb{L}_\infty)}$ is Λ_d-torsion.

Write $Z = Z_A^{(\mathbb{L}_\infty)}$ for brevity, let F_Z be the characteristic power series of Z and write $F_Z = p^{m_0(Z)} \cdot G_Z$, with $p \nmid G_Z$.

(i) *Then $\lambda(Z_A^{(\tilde{K}_\infty)})$ is unbounded in any neighbourhood of K_∞ if the image of G_Z under π_{K_∞} is divisible by p.*

(ii) *On the other hand, suppose that $\pi_{K_\infty}(G_Z)$ is not divisible by p, and that there exists an element $H \in \Lambda_d$ such that $p^s \cdot H$ annihilates Z° for some $s \in \mathbb{N}$, and such that $\pi_{K_\infty}(H)$ is also coprime with p.*
Then $\lambda(Z_A^{(\tilde{K}_\infty)})$ is bounded in a sufficiently small neighbourhood of K_∞.

We stress that the additional assumption in (ii) is automatically satisfied in the case $d = 2$, see Remark 3.6.

Proof. Let $[a_1 : \cdots : a_{d-1}] \in \mathbb{P}^{d-1}(\mathbb{Z}_p)$ correspond to some fixed $\mathbb{Z}_p$-extension $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ of K. It follows from Lemma 3.5 that under our assumptions, $\pi_{K_\infty}(F_Z) \equiv 0 \pmod p$ if and only if $\lambda(Z_{K_\infty})$ is unbounded in any neighbourhood of $[a_1 : \cdots : a_d]$. It follows from Corollary 3.12 or Corollary 3.15 that this is in turn equivalent to $\lambda(Z_A^{(K_\infty)})$ being unbounded on any sufficiently small neighbourhood of K_∞. □

Theorem 3.22. *Let $\mathbb{L}_\infty$ be a $\mathbb{Z}_p^d$-extension of K, and suppose that the decomposition subgroup in $\mathrm{Gal}(\mathbb{L}_\infty/K)$ of each prime v above p is open.*

Let $Z = Z_A^{(\mathbb{L}_\infty)}$, and suppose that Z is a Λ_d-torsion module. For the first part of the theorem, we assume that $Z_A^{(K_\infty)}$ is a Λ-torsion module for each *$\mathbb{Z}_p$-extension $K_\infty \subseteq \mathbb{L}_\infty$ of K, and that the annihilator ideal of the maximal pseudo-null submodule Z° of Z is not contained in any prime ideal of height at most $d-1$. If $Z = X$, then we also assume that A has good ordinary reduction at the primes $v \in \Sigma_p(K)$, and that each such prime ramifies in $\mathbb{L}_\infty$.*

Then the following statements are equivalent:

(i) *the $\lambda(Z_A^{(K_\infty)})$ are bounded as K_∞ runs over the elements from $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$,*
(ii) *$\widehat{l_0}(Z) = 0$.*

Now suppose that $Z_A^{(K_\infty)}$ is not *Λ-torsion for some $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$. Then both statements (i) and (ii) are wrong.*

Proof. Suppose first that $Z_A^{(K_\infty)}$ is Λ-torsion for each K_∞. It follows from Lemma 3.17 that Z is a torsion Λ_d-module. Let $F_Z = p^{m_0(Z)} \cdot G_Z$ be the characteristic power series of Z, as usual. If

$$\widehat{l_0}(F_Z) = \widehat{l_0}(Z_A^{(\mathbb{L}_\infty)}) > 0,$$

then there exist generators $\sigma_1, \ldots, \sigma_d$ of $\mathrm{Gal}(\mathbb{L}_\infty/K)$ and a prime ideal $\mathfrak{p}$ such that

$$G_Z \subseteq \mathfrak{p} \subseteq (p, \sigma_1 - 1, \ldots, \sigma_{d-1} - 1).$$

Then the image of G_Z under the map π_{K_∞} is divisible by p, where

$$K_\infty = \mathbb{L}_\infty^{\langle \sigma_1, \ldots, \sigma_{d-1} \rangle} \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K).$$

In this case, we may deduce from the previous lemma that $\lambda(Z_A^{(\tilde{K}_\infty)})$ is unbounded as $\tilde{K}_\infty$ runs through the elements in a neighbourhood of K_∞.

On the other hand, if $\widehat{l_0}(F_Z) = 0$, then Lemma 3.21 implies that $\lambda(Z_A^{(K_\infty)})$ is locally bounded on $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$. Indeed, since the annihilator ideal of Z° is not contained in any prime ideal of height at most $d-1$, we can deduce the existence of the element $H \in \Lambda_d$ needed in Lemma 3.21 for each $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ as follows (this extends an idea used by Monksy in the proof of [26, Theorem 3.3]). Let

$$J^* = \{w \in \Lambda_d \mid p^s \cdot w \in \mathrm{Ann}(Z^\circ) \text{ for some } s \in \mathbb{N}\}.$$

Then multiplication by p is injective on the quotient Λ_d/J^* by construction. Therefore p is not contained in any minimal prime ideal $\mathfrak{p}$ of J^* by [21, Chapter X, Proposition 2.9] and [7, Theorem 3.1,a.].

Now suppose that $\mathfrak{p}$ is a minimal prime ideal of J^* such that

$$J^* \subseteq \mathfrak{p} \subseteq (p, T_1, \ldots, T_{d-1}),$$

for any choice of variables $T_1, \ldots, T_d$. Then $\mathfrak{p}$ contains also the annihilator ideal of Z°, and therefore the height of $\mathfrak{p}$ must be at least d by our general assumption. But this implies that $\mathfrak{p} = (p, T_1, \ldots, T_{d-1})$. Since we have seen above that p cannot be contained in any minimal prime ideal of J^*, we can conclude that $\mathfrak{p}$ can not be contained in the ideal $(p, T_1, \ldots, T_{d-1})$, for any choice of variables. By the definition of J^*, this proves the existence of an element $H \in \Lambda_d \backslash (p, T_1, \ldots, T_{d-1})$ such that $p^s \cdot H$ annihilates Z° for some sufficiently large $s \in \mathbb{N}$.

We have thus shown that if $\widehat{l_0}(F_Z) = 0$, then $\lambda(Z_A^{(K_\infty)})$ is locally bounded on $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ for each $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$. Since this space is compact (see [8, p. 208]), it follows that the λ-invariant is bounded globally on $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$.

Now we turn to the proof of the last statement of Theorem 3.22. Fix some $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ such that $Z_A^{(K_\infty)}$ is not Λ-torsion. In view of Corollary 3.12(a), respectively, Corollary 3.15(a) this is equivalent to Z_{K_∞} not being Λ-torsion.

Now we use the notion of Fitting ideals. Let π_{K_∞} be the canonical surjection between the Iwasawa algebras. Proposition 2.1(5) implies that we can write $\mathcal{F}_{\Lambda_d}(Z) = (F_Z) \cdot J_Z$, where the ideal J_Z is not contained in any height one prime ideal of Λ_d. Since $\mathcal{F}_{\Lambda_d}(Z)$ is contained in the annihilator ideal of Z and as Z_{K_∞} is not torsion as a Λ-module, we must have $\pi_{K_\infty}(\mathcal{F}_{\Lambda_d}(Z)) = (0)$. We show that this implies that $\pi_{K_\infty}(F_Z) = 0$. Indeed, suppose that this was not true. It follows from the above that there exists some $H \in \Lambda_d$ such that $p^s \cdot H \in \mathrm{Ann}(Z^\circ)$ and $\pi_{K_\infty}(H) \neq 0$ (in fact, $\pi_{K_\infty}(H)$ is not divisible by p). But then Proposition 2.1(3) implies that $\pi_{K_\infty}(\mathcal{F}_{\Lambda_d}(Z)) \neq (0)$.

Therefore $\pi_{K_\infty}(F_Z) = 0$. Writing $F_Z = p^{m_0(Z)} \cdot G_Z$ with G_Z coprime with p, it follows that $\pi_{K_\infty}(G_Z) = 0$. In particular, $\pi_{K_\infty}(G_Z)$ is divisible by p, and therefore $\widehat{l_0}(F_Z) > 0$ from the definition.

On the other hand, it follows from [14, Lemma 4.23] that $\lambda(\pi_{\tilde{K}_\infty}(G_Z))$ is unbounded in a neighbourhood of K_∞. Indeed, let $n \in \mathbb{N}$ be arbitrary. It follows from Lemma 3.1 that we can find a $\mathbb{Z}_p$-extension $\tilde{K}_\infty$ in $\mathcal{E}(K_\infty, n)$

(i.e. the first n layers of K_∞ and $\tilde{K}_\infty$ are equal) such that $Z_{\tilde{K}_\infty}$ is a torsion Λ-module and

$$\mu(Z_{\tilde{K}_\infty}) = m_0(Z) = m_0(F_Z).$$

We claim that this implies that $\pi_{\tilde{K}_\infty}(G_Z)$ is not divisible by p. Indeed, otherwise

$$\mu(\pi_{\tilde{K}_\infty}(F_Z)) > m_0(Z);$$

but

$$(F_{Z_{\tilde{K}_\infty}}) \cdot \mathcal{F}_{\Lambda_1}((Z_{\tilde{K}_\infty})^\circ) = \mathcal{F}_{\Lambda_1}(Z_{\tilde{K}_\infty}) = (\pi_{\tilde{K}_\infty}(F_Z)) \cdot \pi_{\tilde{K}_\infty}(J_Z) \tag{8}$$

in view of Proposition 2.1. Since $(Z_{\tilde{K}_\infty})^\circ$ is a pseudo-null Λ_1-module, the annihilator ideal of this module is not contained in any prime ideal of height one. In view of Proposition 2.1(2), the same holds true for the Fitting ideal $\mathcal{F}_{\Lambda_1}((Z_{\tilde{K}_\infty})^\circ)$. Therefore the equality (8) implies that the characteristic power series $F_{Z_{\tilde{K}_\infty}} \in \Lambda_1$ of $Z_{\tilde{K}_\infty}$ is divisible by $\pi_{\tilde{K}_\infty}(F_Z)$, and thus the above assumption would contradict the fact that

$$\mu(Z_{\tilde{K}_\infty}) = m_0(Z)$$

by the choice of $\tilde{K}_\infty$.

We have shown that $\pi_{\tilde{K}_\infty}(G_Z)$ is not divisible by p. On the other hand,

$$\pi_{K_\infty}(G_Z) \equiv \pi_{\tilde{K}_\infty}(G_Z) \pmod{p, T^{p^n}}.$$

But $\pi_{K_\infty}(F_Z) = 0$ by the above. Therefore the degree of $\pi_{\tilde{K}_\infty}(G_Z)$ must be at least p^n. This concludes the proof of the theorem. □

Remark 3.23.

(1) It follows from Remark 3.9 that the assumption on the annihilator ideal of Z° is also necessary: In the module-theoretic example $M = \Lambda_3/(T_1, T_2 + p)$ given there, the height of the annihilator ideal of $M = M^\circ$ is equal to $2 = d - 1$.

(2) Of course the most important special case is the case where $Z^\circ = \{0\}$. However, this is a strong assumption, in particular for the fine Selmer groups. In fact, by a well-known conjecture of Coates and Sujatha (see [6, Conjecture B], which was formulated for an elliptic curve $A = E$), $Y_A^{(\mathbb{L}_\infty)} = (Y_A^{(\mathbb{L}_\infty)})^\circ$ should be pseudo-null if $\mathbb{L}_\infty$ is any multiple $\mathbb{Z}_p$-extension which contains the cyclotomic $\mathbb{Z}_p$-extension K_∞^{cyc} of K, provided that the fine Selmer group of A over K_∞^{cyc} is cofinitely generated over $\mathbb{Z}_p$. That's why we worked hard in order to extend the theorem to a more general setting.

For Selmer groups, the case $X^\circ = \{0\}$ occurs a little more frequently (cf. also [10] and the proof of Theorem 4.4).

Now we prove a generalisation of the last assertion from Theorem 3.22 which will be used in the next section for the construction of Iwasawa modules with non-trivial $\widehat{l_0}$-invariant.

Theorem 3.24. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension and suppose that $\mathbb{L}_\infty$ contains a $\mathbb{Z}_p^i$-extension K_∞ of K. For brevity, we let $Z = Z_A^{(\mathbb{L}_\infty)}$. We assume that Z is a torsion Λ_d-module. If $Z = X$, then we assume that A has good ordinary reduction at p and that each $v \in \Sigma_p(K)$ ramifies in $\mathbb{L}_\infty$.*

Suppose that $Z_A^{(K_\infty)}$ is not torsion as a Λ_i-module, and that the annihilator ideal of the maximal pseudo-null Λ_d-submodule Z° of Z is not contained in any prime ideal of height at most $d-i$.

Then $\widehat{l_0}(Z) > 0$.

Proof. Let $\pi : \Lambda_d \longrightarrow \Lambda_i$ be the canonical surjection induced by the restriction map

$$\mathrm{Gal}(\mathbb{L}_\infty/K) \twoheadrightarrow \mathrm{Gal}(K_\infty/K),$$

and let $Z_\pi = Z/(\ker(\pi) \cdot Z)$, as usual. It follows from Corollaries 3.13 and 3.16 that the cokernel of the natural map

$$f^\vee : Z_\pi \longrightarrow Z_A^{(K_\infty)}$$

is cofinitely generated over $\mathbb{Z}_p$. In particular, the quotient Z_π is a non-torsion Λ_i-module.

Therefore $\mathcal{F}_{\Lambda_i}(Z_\pi) = (0)$. On the other hand, it follows from Proposition 2.1(4) that

$$\mathcal{F}_{\Lambda_i}(Z_\pi) = \pi(\mathcal{F}_{\Lambda_d}(Z)).$$

We claim that this implies that $\pi(F_Z) = 0$. Indeed, fix generators $\gamma_1, \ldots, \gamma_{d-i}$ of $\mathrm{Gal}(\mathbb{L}_\infty/K_\infty)$ and write $T_j = \gamma_j - 1$ for each such j. Then $\pi(F_Z) = 0$ if and only if

$$F_Z \in (T_1, \ldots, T_{d-i}).$$

On the other hand, since the annihilator ideal of the maximal pseudo-null Λ_d-submodule Z° of Z is not contained in any prime ideal of height $d-i$, it follows that there exists an element $H \in \Lambda$ such that H annihilates the Λ_d-module Z°, and $H \notin (T_1, \ldots, T_{d-i})$. Moreover, if Z can be generated as a Λ_d-module by l elements, then Proposition 2.1(3) implies that

$$H^l \cdot F_Z^l \in \mathcal{F}_{\Lambda_d}(Z).$$

If $\pi(F_Z)$ was non-zero, then this would imply that $\pi(\mathcal{F}_{\Lambda_d}(Z)) \neq (0)$, in contradiction to our assumptions.

Therefore F_Z, and thus also G_Z, lie in the kernel of the map π. This means that

$$G_Z \in (p, T_1, \ldots, T_{d-i}).$$

But then $\widehat{l_0}(G_Z) > 0$ by the definition. □

We conclude the current section with the proof of a natural property concerning the connection between the l_0-invariants and the invariant $\widehat{l_0}$. More precisely, we will consider the following situation. Let $\mathbb{L}_\infty/K$ be a fixed $\mathbb{Z}_p^d$-extension, and consider the set of $\mathbb{Z}_p^2$-extensions L_∞/K with $L_\infty \subseteq \mathbb{L}_\infty$, which we abbreviate to $\mathcal{E}^2$ in what follows. Let Z be either $X_A^{(\mathbb{L}_\infty)}$ or $Y_A^{(\mathbb{L}_\infty)}$. Recall that the Fitting ideal of Z can be written as

$$\mathcal{F}_{\Lambda_d}(Z) = (F_Z) \cdot J_Z,$$

where the ideal J_Z of Λ_d is not contained in any prime ideal of height one.

In the following result we want to relate the following statements to each other:

(1) $l_0(Z_{L_\infty}) = 0$ for all but finitely many $L_\infty \in \mathcal{E}^2$,
(2) $l_0(\pi_{L_\infty}(F_Z)) = 0$ for all but finitely many $L_\infty \in \mathcal{E}^2$ (here $\pi_{L_\infty} : \Lambda_d \longrightarrow \Lambda_2$ and $L_\infty = \mathbb{L}_\infty^{\ker(\pi_{L_\infty})}$, respectively),
(3) $\widehat{l_0}(F_Z) = 0$,
(4) $l_0(\pi_{L_\infty}(F_Z)) = 0$ for **each** π_{L_∞}
(5) $l_0(Z_{L_\infty}) = 0$ for **each** $L_\infty \in \mathcal{E}^2$,
(6) $l_0(Z_A^{(L_\infty)}) = 0$ for **each** $L_\infty \in \mathcal{E}^2$.

Proposition 3.25. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, $d \geq 2$, and let $\mathcal{E}^2$ be as above. We let $Z = X_A^{(\mathbb{L}_\infty)}$ or $Z = Y_A^{(\mathbb{L}_\infty)}$, and we assume that Z is a Λ_d-torsion module.*

(a) *We have implications* (2) $\Rightarrow$ (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (2) *and* (5) $\Rightarrow$ (1). *In particular, the conditions* (2), (3) *and* (4) *are equivalent.*
(b) *If $Z = Y$, then* (5) $\Rightarrow$ (6). *If $Z = X$, then the same holds true, provided that the abelian variety A has good ordinary reduction at the primes of K above p, and that each such prime ramifies in $\mathbb{L}_\infty/K$.*
(c) *Suppose that the annihilator ideal of the maximal pseudo-null Λ_d-submodule Z° of Z is not contained in any prime ideal of height at most d (i.e. we assume Z° to be* finite*). Then* (4) $\Rightarrow$ (5) *and* (2) $\Rightarrow$ (1).

Proof. We start with the proof of (a). It is obvious that statement (5) implies statement (1) and that (4) implies (2).

For the implication (2) $\Rightarrow$ (3), write $F_Z = p^{m_0(Z)} \cdot G_Z$ for some element G_Z which is coprime with p, and suppose that $\widehat{l_0}(F_Z) \neq 0$. Then $v_{\mathfrak{p}}(G_Z) > 0$, where $\mathfrak{p}$ is a prime ideal which is contained in some ideal of the form

$$(p) + \mathrm{Aug}(H).$$

Here $H \cong \mathbb{Z}_p^{d-1}$, and $\mathrm{Aug}(H)$ means the augmentation ideal. In fact, $\mathfrak{p}$ is the pre-image under the canonical projection $\Lambda_d \twoheadrightarrow \Omega_d$ of a minimal prime ideal of $\overline{G_Z}$ which is contained in the image $\overline{\mathrm{Aug}(H)} \subseteq \Omega_d$ of $\mathrm{Aug}(H)$.

Since H contains *infinitely many* rank $d-2$ subgroups H', there exist infinitely many $\mathbb{Z}_p^2$-extensions $L_\infty = \mathbb{L}_\infty^{H'}$ of K such that $l_0(\pi_{L_\infty}(F_Z)) > 0$ for the corresponding maps π_{L_∞}.

Finally, the implication (3) $\Rightarrow$ (4) follows directly from the definitions.

Now we turn to the proof of (b). Under our assumptions we can apply Corollaries 3.13 and 3.16 in order to bound the cokernels of the maps

$$f^\vee : Z_{L_\infty} \longrightarrow Z_A^{(L_\infty)}.$$

Indeed, it follows from these corollaries that the cokernels are finitely generated over $\mathbb{Z}_p$ and thus do not contribute to the l_0-invariant. Therefore the implication (5) $\Rightarrow$ (6) follows from the multiplicity of characteristic power series in short exact sequences of finitely generated torsion Λ_2-modules.

In order to prove (c), suppose that the additional hypothesis on the annihilator ideal of Z° is satisfied. Now suppose that (4) holds, and fix some $L_\infty \in \mathcal{E}^2$. Then $l_0(\pi_{L_\infty}(F_Z)) = 0$. Assume that $l_0(Z_{L_\infty}) > 0$. Then there exists $\sigma \in \mathrm{Gal}(L_\infty/K)$ such that

$$\mathcal{F}_{\Lambda_2}(Z_{L_\infty}) \subseteq (p, \sigma - 1).$$

We write the subgroup of $\mathrm{Gal}(\mathbb{L}_\infty/K)$ fixing L_∞ as $\langle \sigma_1, \ldots, \sigma_{d-2} \rangle$. It follows from the hypothesis in (c) that there exists an annihilator $\tilde{H}$ of Z° which is not contained in the prime ideal

$$\mathfrak{p} := (\sigma_1 - 1, \ldots, \sigma_{d-2} - 1, \sigma - 1, p).$$

Moreover, Proposition 2.1(3) implies that

$$\tilde{H}^l \cdot F_Z^l \in \mathcal{F}_{\Lambda_d}(Z).$$

Since $l_0(\pi_{L_\infty}(F_Z)) = 0$ by (4), we may conclude that the element

$$H := \tilde{H}^l F_Z^l \in \mathcal{F}_{\Lambda_d}(Z)$$

satisfies $H \notin \mathfrak{p}$. This contradicts to the fact that

$$\mathcal{F}_{\Lambda_2}(Z_{L_\infty}) = \pi(\mathcal{F}_{\Lambda_d}(Z)) \subseteq (p, \sigma - 1)$$

and thus $\mathcal{F}_{\Lambda_d}(Z) \subseteq \mathfrak{p}$. □

Corollary 3.26. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, $d \geq 2$, and let $\mathcal{E}^2$ be as above. We assume that Z is Λ_d-torsion. Suppose that the annihilator ideal of the maximal pseudo-null submodule Z° of Z is not contained in any prime ideal of height at most d. If $Z = X$, then we assume that the abelian variety A has good ordinary reduction at the primes of K above p, and that each such prime ramifies in $\mathbb{L}_\infty/K$.*

If $\widehat{l_0}(Z) = 0$, then $l_0(Z_A^{(L_\infty)}) = 0$ for each $L_\infty \in \mathcal{E}^2$.

Proof. Use Proposition 3.25 in order to conclude that

$$(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6).$$

□

4. Applications

We start with an application concerning the *weak Leopoldt conjecture* (see also Theorem 4.2 below). Since the underlying proof works both for fine Selmer groups and for Selmer groups, we will formulate the key result for both these objects simultaneously.

Theorem 4.1. *In the following, we let Z be either X or Y (the choice of Z is fixed throughout the theorem).*

Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, and suppose that $\mathrm{rank}_\Lambda(Z_A^{(L_\infty)}) = 0$ *for some $\mathbb{Z}_p$-extension $L_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ of K. If $Z = X$, then we assume that A has good ordinary reduction at each prime $v \in \Sigma_p(K)$ and that each such prime ramifies in $L_\infty \subseteq \mathbb{L}_\infty$. If $d > 2$, then we assume that the annihilator ideal of the maximal pseudo-null submodule Z° of Z is not contained in any prime ideal of height at most d.*

(a) *Suppose that $d = 2$. Then* $\mathrm{rank}_\Lambda(Z_A^{(K_\infty)}) = 0$ *for all but finitely many $\mathbb{Z}_p$-extensions $K_\infty \subseteq \mathbb{L}_\infty$ of K.*
(b) *Let now $d \in \mathbb{N}$ be arbitrary again. If, in addition,*

$$\widehat{l_0}(Z_A^{(\mathbb{L}_\infty)}) = 0,$$

then $\mathrm{rank}_\Lambda(Z_A^{(K_\infty)}) = 0$ *holds for* each $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$.

Proof. For the first statement, we note that it follows from Lemma 3.17 that $Z_A^{(\mathbb{L}_\infty)}$ is a torsion Λ_2-module. By Lemma 3.1, the quotient $(Z_A^{(\mathbb{L}_\infty)})_{K_\infty}$ is a torsion Λ-module for all but finitely many $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$. On the other hand, our control Theorems 3.11 and 3.14 imply that the cokernels of the maps

$$f^\vee\colon (Z_A^{(\mathbb{L}_\infty)})_{K_\infty} \longrightarrow Z_A^{(K_\infty)}$$

are Λ-torsion for all $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$. This proves (a).

For (b), we first consider the case $d = 2$. Let $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ be arbitrary. We write $\Lambda_2 = \mathbb{Z}_p[[S, T]]$, where $\mathrm{Gal}(\mathbb{L}_\infty/K_\infty) = \langle T + 1\rangle$, and we abbreviate $Z_A^{(\mathbb{L}_\infty)}$ to Z.

Let $F_Z \in \Lambda_2$ denote the characteristic power series of Z. As in the proof of [26, Theorem 3.3], we can choose an element $H \in \Lambda_2$ such that the image $\overline{H}$ of H in $\Omega_2 = \Lambda_2/p$ is not divisible by $\overline{T}$ and $p^{m_0(Z)+s} \cdot F_Z \cdot H$ annihilates Z for sufficiently large $s \in \mathbb{N}$. Since $l_0(Z) = 0$ by assumption, $\overline{F_Z} \in \Omega_2$ is also coprime with $\overline{T}$. Therefore the same holds true for $\overline{F_Z \cdot H}$, and $Z_{K_\infty} = Z/(T \cdot Z)$ is a torsion $\Lambda = \mathbb{Z}_p[[S]]$-module. In view of Lemmas 3.11 and 3.14 the same holds true for $Z_A^{(K_\infty)}$.

Now let $d \in \mathbb{N}$ be arbitrary, and let $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$. We consider the $\mathbb{Z}_p^2$-extension

$$\mathbb{L}_\infty^{(2)} := K_\infty \cdot L_\infty$$

of K, where L_∞ is the $\mathbb{Z}_p$-extension of K from the hypotheses of the theorem. It follows from Corollary 3.26 that $l_0(Z_A^{(\mathbb{L}_\infty^{(2)})}) = 0$. Therefore the first part of the proof of (b) implies that $\mathrm{rank}_\Lambda(Z_A^{(K_\infty)}) = 0$. □

Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, and let A be an abelian variety defined over K. Recall from Remark 3.19 that $Y_A^{(\mathbb{L}_\infty)}$ being a torsion Λ_d-module is known as the weak Leopoldt conjecture holds for A over $\mathbb{L}_\infty$. Therefore we may derive from Theorem 4.1 the following

Theorem 4.2. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, and suppose that the weak Leopoldt conjecture holds for some $\mathbb{Z}_p$-extension $L_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ of K. If $d > 2$, then we assume that the annihilator ideal of the maximal pseudo-null submodule of $Y = Y_A^{(\mathbb{L}_\infty)}$ is not contained in any prime ideal of height less than d.*

(a) *If $d = 2$, then the weak Leopoldt conjecture holds for A over all but finitely many $\mathbb{Z}_p$-extensions of K which are contained in $\mathbb{L}_\infty$.*

(b) *Let $d \geq 2$ be arbitrary. If, in addition, $\widehat{l_0}(Y_A^{(\mathbb{L}_\infty)}) = 0$, then the weak Leopoldt conjecture holds in fact for* all *$\mathbb{Z}_p$-extensions $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ of K.*

Remark 4.3. In [23], Lim proved a special instance of this result. To be more precise, Lim considered abelian varieties over $\mathbb{Z}_p^2$-extensions of K which contain the cyclotomic $\mathbb{Z}_p$-extension K_∞^{cyc} of K. The assertion (a) of Theorem 4.2 then follows from [23, Theorem 3.9], and Lim proved the assertion of (b) under the potentially stronger assumption that $Y_A^{(\mathbb{L}_\infty)}$ is pseudo-null over Λ_2 (see [23, Proposition 3.8]).

In the case of Selmer groups, we can actually go one step farther.

Theorem 4.4. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, and suppose that*

(i) $\mathrm{rank}_\Lambda(X_A^{(L_\infty)}) = 0$ *for some $L_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ such that each $v \in \Sigma_p(K)$ ramifies in L_∞,*

(ii) *for any $v \in \Sigma_p(K)$ its decomposition subgroup of* $\mathrm{Gal}(\mathbb{L}_\infty/K)$ *is open,*

(iii) *A has good ordinary reduction at each $v \in \Sigma_p(K)$,*

(iv) *K is totally imaginary, and*

(v) *$A(\mathbb{L}_\infty)[p^\infty]$ is finite.*

If $d > 2$, then we moreover assume that the annihilator ideal of the maximal pseudo-null Λ_d-submodule of $X_A^{(\mathbb{L}_\infty)}$ is not contained in any prime ideal of height at most d. Then the following statements hold.

(a) *The following are equivalent:*

  (1) $\widehat{l_0}(X_A^{(\mathbb{L}_\infty)}) = 0$,

  (2) *$X_A^{(K_\infty)}$ is Λ-torsion for each $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$, and $\mu(X_A^{(K_\infty)})$ is constant as one runs over the $\mathbb{Z}_p$-extensions $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ of K.*

(3) $\lambda(X_A^{(K_\infty)})$ *is bounded as* K_∞ *runs over the* $\mathbb{Z}_p$*-extensions of* K *which are contained in* $\mathbb{L}_\infty$*.*

(b) *In particular, if* $\widehat{l_0}(X_A^{(\mathbb{L}_\infty)}) = 0$ *and* $\mu(X_A^{(K_\infty)}) = 0$ *holds for* any $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$*, then the same holds in fact true for* every $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$*.*

Proof. We start with the proof of (a).

(1) $\Longrightarrow$ (2): Suppose that $\widehat{l_0}(X_A^{(\mathbb{L}_\infty)}) = 0$. Then Theorem 4.1 implies that $X_A^{(K_\infty)}$ is Λ-torsion for each $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$. We will show that $\mu(X_A^{(K_\infty)}) = \mu(X_A^{(L_\infty)})$ for each $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$. To this purpose, let $\tilde{K}_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ be arbitrary, and consider the $\mathbb{Z}_p^2$-extension

$$\mathbb{L}_\infty^{(2)} = L_\infty \cdot \tilde{K}_\infty$$

of K, where L_∞ is from hypothesis (i) of the theorem.

It follows from Corollary 3.26 and Theorem 3.22 that $\lambda(X_A^{(K_\infty)})$ is bounded on $\mathcal{E}^{\subseteq \mathbb{L}_\infty^{(2)}}(K)$; let C be an upper bound. The additional hypothesis that $A(\mathbb{L}_\infty)[p^\infty]$ is finite implies that $A(K_\infty)[p^\infty]$ is finite and of bounded order as K_∞ runs over the elements of $\mathcal{E}^{\subseteq \mathbb{L}_\infty^{(2)}}(K)$. It follows from the main result of [10] (see [20, Theorem 3.4] for more details) that the cardinality of the maximal finite Λ-submodule $(X_A^{(K_\infty)})^\circ$ of $X_A^{(K_\infty)}$ is bounded by some constant $\tilde{C} \in \mathbb{N}$ as K_∞ runs over the elements in $\mathcal{E}^{\subseteq \mathbb{L}_\infty^{(2)}}(K)$.

Since $X_A^{(K_\infty)}$ is Λ-torsion for each $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$, the quotient $X_A^{(K_\infty)}/f$ will be finite for every $f \in \Lambda$ which is coprime with the characteristic power series of $X_A^{(K_\infty)}$. We make a concrete choice: We choose n large enough such that

$$\tilde{C} < n \quad \text{and} \quad n \cdot C + \tilde{C} < p^{n+1} - p^n,$$

and we consider the polynomial

$$\nu_{2n,n} = \frac{\omega_{2n}}{\omega_n} \in \mathbb{Z}_p[T],$$

where $\omega_n = \omega_n(T) = (T+1)^{p^n} - 1$. The polynomial $\nu_{2n,n}$ is a product of irreducible factors of degrees equal to $p^{i+1} - p^i$ for $i \in \{n, \ldots, 2n-1\}$. In particular, since $\lambda(X_A^{(K_\infty)}) < p^{n+1} - p^n$ for each $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty^{(2)}}(K)$ by the choice of n, the characteristic power series of each $X_A^{(K_\infty)}$ is coprime with each of the irreducible factors of $\nu_{2n,n}$, and therefore also coprime with $\nu_{2n,n}$. Therefore

$$\operatorname{rank}_{\nu_{2n,n}}(X_A^{(K_\infty)}) := v_p(|X_A^{(K_\infty)}/\nu_{2n,n} X_A^{(K_\infty)}|)$$

is finite for every $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$.

It follows from Lemma 3.1 and Corollary 3.15 that

$$\mu(X_A^{(L)}) = m_0(X_A^{(\mathbb{L}_\infty^{(2)})})$$

for all but finitely many $L \in \mathcal{E}^{\subseteq \mathbb{L}_\infty^{(2)}}(K)$ (indeed: Using the notation from Corollary 3.15(a), if $v \in \Sigma(K) \backslash \Sigma_s(K)$, i.e. v is not completely split in the $\mathbb{Z}_p^2$-extension $\mathbb{L}_\infty^{(2)}$ of K, then the $\mathbb{Z}_p$-rank of the decomposition group is at least one, i.e. there exists at most one $\mathbb{Z}_p$-extension L of K in $\mathbb{L}_\infty^{(2)}$ such that v splits completely in L). Suppose that $\mu(X_A^{(K_\infty)}) \neq m_0(X_A^{(\mathbb{L}_\infty^{(2)})})$ for some fixed K_∞. Since $A(\mathbb{L}_\infty)[p^\infty]$ is finite by assumption, it follows that $A(K_\infty)[p^\infty]$ is finite. Therefore we can apply [16, Theorem 4.5 and Corollary 3.8] in order to conclude that there exists a Greenberg neighbourhood $U = \mathcal{E}(K_\infty, m) \cap \mathcal{E}^{\subseteq \mathbb{L}_\infty^{(2)}}(K)$ of K_∞ such that

$$\mathrm{rank}_{\nu_{2n,n}}(X_A^{(L)}) = \mathrm{rank}_{\nu_{2n,n}}(X_A^{(K_\infty)}) \tag{9}$$

for each $L \in U$, with the above choice of $\nu_{2n,n}$.

We may assume that m has been chosen large enough to ensure that

$$\mu(X_A^{(L)}) = m_0(X_A^{(\mathbb{L}_\infty^{(2)})})$$

holds for each $L \in U$ which is different from K_∞. It is a general fact that

$$\mathrm{rank}_{\nu_{2n,n}}(X_A^{(L)}) = \mathrm{rank}_{\nu_{2n,n}}(E_{X_A^{(L)}}) + \mathrm{rank}_{\nu_{2n,n}}((X_A^{(L)})^\circ) \tag{10}$$

for each L (see the proof of [15, Theorem 3.10(iii)]). Since $|(X_A^{(L)})^\circ| \leq \tilde{C}$ is bounded as L runs over the elements in $U \subseteq \mathcal{E}^{\subseteq \mathbb{L}_\infty^{(2)}}(K)$, we may conclude that

$$|\mathrm{rank}_{\nu_{2n,n}}(E_{X_A^{(L)}}) - \mathrm{rank}_{\nu_{2n,n}}(E_{X_A^{(K_\infty)}})| \leq \tilde{C} \tag{11}$$

for every $L \in U$. Now

$$\mathrm{rank}_{\nu_{2n,n}}(E_{X_A^{(K_\infty)}}) = \mu(X_A^{(K_\infty)}) \cdot (p^{2n} - p^n) + n \cdot \lambda(X_A^{(K_\infty)}),$$

and an analogous formula holds for L. Indeed, the $\nu_{2n,n}$-rank of the 'λ'-part $\bigoplus_{j=1}^t \Lambda/(g_j^{n_j})$ of $E_{X_A^{(K_\infty)}}$ (i.e. the sum over all the $\mathbb{Z}_p$-free quotients occuring in $E_{X_A^{(K_\infty)}}$) is given by

$$\sum_\zeta v_p(F(\zeta - 1)),$$

where $F = \prod_{j=1}^t g_j^{n_j}$, and where ζ runs over the roots of unity of exact orders p^i, $i \in \{n+1, \ldots, 2n\}$, which are contained in some fixed algebraic closure of $\mathbb{Q}_p$. Since $\deg(F) < p^{n+1} - p^n$ by the choice of n, it follows that

$$v_p(F(\zeta - 1)) = \frac{\deg(F)}{p^{i+1} - p^i} = \frac{\lambda(X_A^{(K_\infty)})}{p^{i+1} - p^i}$$

for each ζ of exact order p^{i+1}, because $\lambda(X_A^{(K_\infty)}) = \deg(F)$ by the definition. A similar formula holds for $\mathrm{rank}_{\nu_{2n,n}}(E_{X_A^{(L)}})$, $L \in U$ arbitrary, since

$$\lambda(X_A^{(L)}) \leq C < p^{n+1} - p^n$$

by our choices of C and n.

Therefore inequality (11) implies that

$$|\left(\mu(X_A^{(K_\infty)}) - \mu(X_A^{(L)})\right) \cdot (p^{2n} - p^n) + n \cdot \left(\lambda(X_A^{(K_\infty)}) - \lambda(X_A^{(L)})\right)| \leq \tilde{C}.$$

In view of the choice of n, this shows that the μ-invariants must be equal, i.e. the assumption $\mu(X_A^{(K_\infty)}) \neq m_0(X_A^{(\mathbb{L}_\infty^{(2)})})$ cannot hold. Moreover, we may conclude that

$$\lambda(X_A^{(K_\infty)}) = \lambda(X_A^{(L)})$$

for each $L \in U \subseteq \mathcal{E}^{\subseteq \mathbb{L}_\infty^{(2)}}(K)$, since $n > \tilde{C}$.

(2) $\Longrightarrow$ (3): Now we assume that $\mu(X_A^{(K_\infty)}) = \mu(X_A^{(L_\infty)})$ for each $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$. Then [16, Theorem 4.11] implies that for each $K_\infty \in \mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ there exists a neighbourhood $\mathcal{E}(K_\infty, n)$ such that

$$\lambda(X_A^{(L)}) \leq \lambda(X_A^{(K_\infty)})$$

for each $L \in \mathcal{E}(K_\infty, n)$, i.e. the λ-invariants are locally bounded. Since the set $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$ is compact with respect to Greenberg's topology, we see that the fact that μ is constant implies that λ is bounded on $\mathcal{E}^{\subseteq \mathbb{L}_\infty}(K)$.

(3) $\Longrightarrow$ (1): This follows from Theorem 3.22.

Therefore we have proven statement (a). Assertion (b) is a direct special case. □

In [18], we have constructed families of elliptic curves E over imaginary-quadratic number fields K (see Theorem 4.5 for the details) such that $\lambda(X_E^{(K_\infty)})$ is *unbounded* as K_∞ runs over the $\mathbb{Z}_p$-extensions of K (note that the composite of all the $\mathbb{Z}_p$-extensions of an imaginary-quadratic field K is a $\mathbb{Z}_p^2$-extension $\mathbb{L}_\infty$ of K). In this example, E had good ordinary reduction at the primes above p. Therefore the results from Sect. 3 are applicable, and we may deduce the following

Theorem 4.5. *Let E be an elliptic curve of conductor N defined over $\mathbb{Q}$, and let K be an imaginary quadratic field such that $\mathcal{O}_K^\times = \{\pm 1\}$. We assume that*

1. *the rational prime p does not divide $6N\mathrm{disc}(K)h_K|E/E^0|$, where h_K denotes the class number of K, $\mathrm{disc}(K)$ denotes the discriminant, and E/E^0 means the set of connected components of the Néron model of E over $\mathrm{Spec}(\mathcal{O}_K)$,*
2. *E has good ordinary reduction at each $v \in \Sigma_p(K)$,*
3. *the Galois representation*

$$\rho_p \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(E[p^\infty])$$

   is surjective,
4. *$E(k_v)[p^\infty] = \{0\}$ for each $v \in \Sigma_p(K)$ (here k_v denotes the finite residue field at v),*
5. *$|E(\mathbb{F}_p)| \not\equiv -1 \pmod p$ if p splits in $K/\mathbb{Q}$, and*

(6) *every prime $q \mid N$ splits in $K/\mathbb{Q}$.*

Then $l_0(X_E^{(\mathbb{K}_\infty)}) > 0$, where $\mathbb{K}_\infty$ denotes the composite of all $\mathbb{Z}_p$-extensions of K.

Proof. The hypotheses (1)–(6) ensure that $X_E^{(K_\infty^a)}$ is non-torsion as a Λ-module, where K_∞^a denotes the anticyclotomic $\mathbb{Z}_p$-extension of K.

Since E has good ordinary reduction at the primes above p, it follows from the results of Kato and Rohrlich that $X_E^{(K_\infty^c)}$ is Λ-torsion, where K_∞^c denotes the cyclotomic $\mathbb{Z}_p$-extension of K. Since each prime of K above p ramifies in K_∞^c/K, it follows from Lemma 3.17 that $X_E^{(\mathbb{K}_\infty)}$ is a torsion Λ_2-module. The result follows from Theorem 4.1.

Alternatively, it has been shown in [18, Corollary 1.2] that $\lambda(X_E^{(K_\infty)})$ is unbounded as K_∞ runs over the $\mathbb{Z}_p$-extensions of K, provided that $\mu(X_E^{(K_\infty^c)}) = 0$. Therefore, under this additional hypothesis, Theorem 3.22 can be applied to deduce the result (see also Remark 3.6). □

In [18, Example 7.14], we have given a concrete example where all the hypotheses from this theorem are satisfied: Let $K = \mathbb{Q}(\sqrt{-7})$, and consider the elliptic curve E defined by

$$E\colon y^2 + y = x^3 - x^2 - 10x - 20.$$

Then the hypotheses from Theorem 4.5 are satisfied for the two primes 37 and 43, both of which are split in $K/\mathbb{Q}$. To the authors' knowledge, this provides the first known example at all of an Iwasawa module having a non-trivial l_0-invariant.

Remark 4.6. In the situation of Theorem 4.5, the Iwasawa module $X_E^{(\mathbb{K}_\infty)}$ is Λ_2-torsion. Theorem 4.1 implies that $X_E^{(K_\infty)}$ is Λ-torsion for all but finitely many $\mathbb{Z}_p$-extensions K_∞ of K. Therefore Theorem 3.20 implies that the *μ-invariants* of the Iwasawa modules $X_E^{(K_\infty)}$ are bounded as K_∞ runs over the $\mathbb{Z}_p$-extensions of K. One can in fact say more: If $\mu(X_E^{(K_\infty^c)}) = 0$ in the setting of Theorem 4.5, and as $X_E^{(K_\infty^c)}$ is a Λ-torsion module by the results of Kato and Rohrlich, [16, Theorem 1.1] implies that the λ-invariants $\lambda(X_E^{(K_\infty)})$ are bounded as K_∞ runs over the $\mathbb{Z}_p$-extensions of K which are contained in some sufficiently small neighbourhood of K_∞^c. Now the main result of [20] (more precisely, Proposition 2.6 and the implication $(e) \implies (b)$ from Theorem 1.1 of loc.cit.) implies that $m_0(X_E^{(\mathbb{K}_\infty)}) = 0$. We may deduce from Lemma 3.1 and Corollary 3.15(a) and (b) that $\mu(X_E^{(K_\infty)}) = 0$ for all but finitely many $K_\infty \in \mathcal{E}^{\subseteq \mathbb{K}_\infty}(K)$. Note that we cannot apply Theorem 4.4, since $\mathrm{rank}_\Lambda(X_A^{(K_\infty^a)}) > 0$.

Now we derive from Theorem 4.5 the existence of a $\mathbb{Z}_p^d$-extension $\mathbb{L}_\infty/L$, $d > 2$, such that $\widehat{l_0}(X_E^{(\mathbb{L}_\infty)}) \neq 0$. To this purpose, we first enlarge the base field, since the imaginary quadratic number field K has not enough $\mathbb{Z}_p$-extensions.

Lemma 4.7. *Let E, p and $\mathbb{K}_\infty/K$ be as in Theorem 4.5, and let L/K be a finite normal p-extension with Galois group H. We assume that $\mu(X_E^{(K^c_\infty)}) = 0$, where K^c_∞ denotes the cyclotomic $\mathbb{Z}_p$-extension of K. We assume that $L \cap \mathbb{K}_\infty = K$. Let $L_\infty = \mathbb{K}_\infty \cdot L$. Then $l_0(X_E^{(L_\infty)}) \neq 0$.*

Proof. As we have seen in Remark 4.6, we can choose a neighbourhood $U = \mathcal{E}(K^a_\infty, n)$ of the anticyclotomic $\mathbb{Z}_p$-extension of K such that

$$\mathrm{rank}_\Lambda(X_E^{(K_\infty)}) = \mu(X_E^{(K_\infty)}) = 0$$

for each $K^a_\infty \neq K_\infty \in U$. Moreover, since each prime above p ramifies in K^a_∞, we may assume that U has been chosen small enough to ensure that the same is true for each $K_\infty \in U$. Finally, by hypothesis (6) of Theorem 4.5, each prime of K of bad reduction splits in $K/\mathbb{Q}$. It then follows from [3, Theorem 2] that each such prime is finitely split in the anticyclotomic extension K^a_∞ of K. Since $p \neq 2$, we can assume that $\Sigma \backslash \Sigma_p$ contains exactly the primes of bad reduction, and we may thus assume that each $v \in \Sigma$ is finitely split in every $K_\infty \in U$.

To each such K_∞, we consider the finite p-extension $K'_\infty = K_\infty \cdot L$ of K_∞, and we identify $\mathrm{Gal}(K'_\infty/K_\infty)$ with H.

Now we consider the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Sel}(E/K_\infty) & \longrightarrow & H^1(K_\Sigma/K_\infty, E[p^\infty]) & \longrightarrow & \prod_{v\in S} J_v(E/K_\infty) \\
& \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
0 \longrightarrow & \mathrm{Sel}(E/K'_\infty)^H & \longrightarrow & H^1(K_\Sigma/K'_\infty, E[p^\infty])^H & \longrightarrow & \prod_{v\in S} J_v(E/K'_\infty),
\end{array}$$

where $J_v(E/K_\infty) = \varinjlim \bigoplus_{w|v} H^1(F_w, E)[p^\infty]$ and the limit is taken over all finite subextensions F_w/K_v of $K_{\infty,w}$. Moreover, $J_v(E/K'_\infty)$ is defined in an analogous manner.

We want to show that the Pontryagin duals of $\ker(\beta)$, $\mathrm{coker}(\beta)$ and $\ker(\gamma)$ are finite, since this will show that also $\ker(\alpha)$ and $\mathrm{coker}(\alpha)$ are finite.

Since $\ker(\beta) = H^1(H, E(K'_\infty)[p^\infty])$ and $\mathrm{coker}(\beta) = H^2(H, E(K'_\infty)[p^\infty])$ are both cofinitely generated over $\mathbb{Z}_p$ and annihilated by some power of p (the latter holds true since H is finite), their Pontryagin duals are also annihilated by the same power of p. Therefore $\ker(\beta)^\vee$ and $\mathrm{coker}(\beta)^\vee$ are finite.

Now we consider the kernel of γ. Since each prime of K above p ramifies in the $\mathbb{Z}_p$-extension K_∞, both K_∞/K and K'_∞/K' are deeply ramified extensions in the sense of Coates and Greenberg (see [4, Theorem 2.13]). Therefore the kernel of γ may be written in the form $C \times D$, where

$$C = \prod_{w\in\Sigma(K_\infty)\backslash\Sigma_p(K_\infty)} H^1(H, E(K'_{\infty,w})[p^\infty])$$

and

$$D = \prod_{w\in\Sigma_p(K_\infty)} H^1(H_w, \tilde{E}(k'_{\infty,w})[p^\infty]).$$

Here we have fixed, by abuse of notation, a prime w of K'_∞ above $w \in \Sigma(K_\infty)$, respectively, and $H_w \subseteq H$ denotes the corresponding decomposition subgroup. Moreover, $k'_{\infty,w}$ denotes the residue field of $K'_{\infty,w}$ and $\tilde{E}$ is the reduction of E over the residue field.

Again, each of these cohomology groups is cofinitely generated over $\mathbb{Z}_p$ and annihilated by a power of p and hence is finite. Since no prime $v \in \Sigma$ is totally split in K_∞/K, it follows that $\Sigma(K_\infty)$ is finite, and therefore the Pontryagin dual of the kernel of γ is finite.

We have shown that the kernel and the cokernel of the natural dualised map

$$\psi^{(K_\infty)}\colon (X_E^{(K'_\infty)})_H \longrightarrow X_E^{(K_\infty)}$$

are finite. Recall that $\mathrm{rank}_\Lambda(X_E^{(K_\infty)}) = \mu(X_E^{(K^c_\infty)}) = 0$ for each $K^a_\infty \neq K_\infty \in U$. Therefore $\mathrm{rank}_p(X_E^{(K_\infty)})$ is finite for each such K_∞, and it follows that the same holds for $\mathrm{rank}_p((X_E^{(K'_\infty)})_H)$. Nakayama's Lemma implies that $X_E^{(K'_\infty)}$ is Λ-torsion and has μ-invariant zero. Since $\mathrm{rank}_\Lambda(X_E^{(K^a_\infty)}) > 0$ by our hypotheses, the proof of the last assertion of Theorem 3.22 shows that the λ-invariants $\lambda(X_E^{(K_\infty)})$ are unbounded as K_∞ runs over the elements in U. Since the cokernel of $\psi^{(K_\infty)}$ is finite for each K_∞, and as

$$\lambda(X_E^{(K'_\infty)}) \geq \mathrm{rank}_{\mathbb{Z}_p}((X_E^{(K'_\infty)})_H),$$

it follows that $\lambda(X_E^{(K'_\infty)})$ is also unbounded. But then Theorem 3.22 implies that $l_0(X_E^{(L_\infty)}) > 0$ (recall that the additional hypothesis on the maximal pseudo-null submodule of $X_E^{(L_\infty)}$ is not needed in the $d = 2$ case, as has been pointed out in Remark 3.6). □

The following theorem follows if we combine the previous Lemma 4.7 with Corollary 3.26.

Theorem 4.8. *Let E be an elliptic curve defined over $\mathbb{Q}$, and let K be an imaginary quadratic number field such that the hypotheses from Theorem 4.5 are satisfied. We assume that $\mu(X_E^{(K^c_\infty)}) = 0$, where K^c_∞ denotes the cyclotomic $\mathbb{Z}_p$-extension of K.*

Moreover, let L/K be any finite normal p-extension, and let $\mathbb{L}_\infty$ denote the composite of all $\mathbb{Z}_p$-extensions of L. Let $d = \mathrm{rank}_{\mathbb{Z}_p}(\mathrm{Gal}(\mathbb{L}_\infty/K))$.

We assume that the annihilator ideal of the maximal pseudo-null Λ_d-submodule of $X_E^{(\mathbb{L}_\infty)}$ is not contained in any prime ideal of height at most d.

Then $\widehat{l_0}(X_E^{(\mathbb{L}_\infty)}) > 0$.

We mention one final application. We want to construct examples of Iwasawa modules with non-trivial $\widehat{l_0}$-invariant. Since it is difficult to check the condition on the annihilator ideal of the maximal pseudo-null submodule of

$X_E^{(\mathbb{L}_\infty)}$, we choose a different approach. In the following, let A be an arbitrary abelian variety defined over K, let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, and write Z for either $Y_A^{(\mathbb{L}_\infty)}$ or $X_A^{(\mathbb{L}_\infty)}$ (i.e. the following result holds for both Selmer groups and fine Selmer groups).

Theorem 4.9. *Let $\mathbb{L}_\infty/K$ be a $\mathbb{Z}_p^d$-extension, and suppose that Z is a torsion Λ_d-module, and that $\mathbb{L}_\infty$ contains a $\mathbb{Z}_p^{d-1}$-extension K_∞ of K such that $Z_A^{(K_\infty)}$ is a non-torsion Λ_{d-1}-module. Then $\widehat{l_0}(Z) > 0$.*

Proof. This follows from the $i = d-1$ special case of Theorem 3.24, since the annihilator ideal of the maximal pseudo-null Λ_d-submodule Z° of Z is not contained in any prime ideal of Λ_d of height one. □

Now we state our main application of this result. We mention only one example for the scope of Theorem 4.9; the base field K could be chosen in many different ways.

Corollary 4.10. *Let $K = \mathbb{Q}(\zeta_{3^i})$ for some $i \geq 2$, where ζ_{3^i} denotes a primitive 3^ith root of unity (in some fixed algebraic closure of $\mathbb{Q}$). Let E be an elliptic curve defined over $\mathbb{Q}$.*

We assume that $p \equiv 2 \pmod 3$ is a prime number such that E has good ordinary reduction at p, and that the conductor N_E of E satisfies $N_E \equiv 1 (\mathrm{mod}\ 3)$.

If $E(K)[p] = \{0\}$, and if p divides none of the local Tamagawa factors c_v of the primes v of K of bad reduction, then there exists a $\mathbb{Z}_p^{3^{i-1}+1}$-extension $\mathbb{L}_\infty$ of K such that the $\widehat{l_0}$-invariant of $X_E^{(\mathbb{L}_\infty)}$ is non-trivial.

Proof. Let $k \subseteq K$ be the unique imaginary quadratic subfield. It follows from [27, Example 5.1] that there exists a $\mathbb{Z}_p^{r_2(K)}$-extension K_∞ of K such that complex conjugation acts by -1 on $\mathrm{Gal}(K_\infty/K)$. Note that

$$r_2(K) = [K : \mathbb{Q}]/2 = 3^{i-1}.$$

In view of our hypotheses, [27, Corollary 3.6 and Example 5.1] imply that $X_E^{(K_\infty)}$ is a non-torsion $\Lambda_{r_2(K)}$-module. More precisely, the condition (c) from [27, Corollary 3.6] holds in view of the two facts $4 \nmid [K : \mathbb{Q}]$ and $p \equiv 2 \pmod 3$ (the latter implies that p has even order in $(\mathbb{Z}/3\mathbb{Z})^\times$), as is explained in [27, Example 5.1]. If χ denotes the quadratic character of $k = \mathbb{Q}(\sqrt{-3})$, then $\chi(N_E) = 1$ in view of the condition $N_E \equiv 1 \pmod 3$.

Now let $\mathbb{L}_\infty := K_\infty \cdot K_{cyc}$ be the compositum with the cyclotomic $\mathbb{Z}_p$-extension of K. Since K is a totally abelian field, it follows from the results of Kato and Rohrlich (see [12,30]) that $X_A^{(K_{cyc})}$ is a torsion Λ_1-module. Therefore $X_A^{(\mathbb{L}_\infty)}$ is Λ_d-torsion in view of Lemma 3.17, where $d = r_2(K) + 1$. Then it follows from Theorem 4.9 that $\widehat{l_0}(X_A^{(\mathbb{L}_\infty)}) > 0$. □

Example 4.11. Let E be the elliptic curve with Cremona label 19a1, and let $K = \mathbb{Q}(\zeta_9)$. Computations in SAGE [31] verify that E has good ordinary reduction at $p = 5$, $E(K)[5] = \{0\}$ and that 5 divides none of the Tamagawa numbers c_v of the primes v of K. By the previous corollary there exists a $\mathbb{Z}_p^4$-extension $\mathbb{L}_\infty$ of K such that the $\widehat{l_0}$-invariant of $X_E^{(\mathbb{L}_\infty)}$ is non-trivial.

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References

[1] Babaĭcev, V.A.: On the boundedness of Iwasawa's μ-invariant. Izv. Akad. Nauk SSSR Ser. Mat. **44**(1), 3–23 (1980). (**238**)

[2] Bosch, Siegfried: Algebra. Springer-Lehrb, 6th edn. Springer, Berlin (2006)

[3] Brink, David: Prime decomposition in the anti-cyclotomic extension. Math. Comput. **76**(260), 2127–2138 (2007)

[4] Coates, J., Greenberg, R.: Kummer theory for abelian varieties over local fields. Invent. Math. **124**(1–3), 129–174 (1996)

[5] Cuoco, Albert A., Monsky, Paul: Class numbers in Z_p^d-extensions. Math. Ann. **255**(2), 235–258 (1981)

[6] Coates, J., Sujatha, R.: Fine Selmer groups of elliptic curves over p-adic Lie extensions. Math. Ann. **331**(4), 809–839 (2005)

[7] Eisenbud, David: Commutative Algebra. With a View Toward Algebraic Geometry. Springer, Berlin (1995)

[8] Greenberg, Ralph: The Iwasawa invariants of **Γ**-extensions of a fixed number field. Am. J. Math. **95**, 204–214 (1973)

[9] Greenberg, Ralph: Galois theory for the Selmer group of an abelian variety. Compos. Math. **136**, 255–297 (2003)

[10] Hachimori, Yoshitaka, Matsuno, Kazuo: On finite Λ-submodules of Selmer groups of elliptic curves. Proc. Am. Math. Soc. **128**(9), 2539–2541 (2000)

[11] Iwasawa, Kenkichi: On Z_l-extensions of algebraic number fields. Ann. Math. **2**(98), 246–326 (1973)

[12] Kato, K.: p-adic Hodge theory and values of zeta functions of modular forms. In: Cohomologies p-adiques et applications arithmétiques (III), pp. 117–290. Paris: Société Mathématique de France (2004)

[13] Kim, Chan-Ho., Kurihara, Masato: On the refined conjectures on Fitting ideals of Selmer groups of elliptic curves with supersingular reduction. Int. Math. Res. Not. IMRN **14**, 10559–10599 (2021)

[14] Kleine, Sören: A New Approach for the Investigation of Iwasawa Invariants. Ph.D. thesis, University of Göttingen (2015)

[15] Kleine, Sören.: Local behavior of Iwasawa's invariants. Int. J. Number Theory **13**(4), 1013–1036 (2017)

[16] Kleine, Sören.: Bounding the Iwasawa invariants of Selmer groups. Can. J. Math. **73**(5), 1390–1422 (2021)

[17] Kleine, Sören.: Generalised Iwasawa invariants and the growth of class numbers. Forum Math. **33**(1), 109–127 (2021)

[18] Kleine, Sören.: On the Iwasawa invariants of non-cotorsion Selmer groups. Asian J. Math. **26**(3), 373–406 (2022)

[19] Kleine, S., Müller, K.: Fine Selmer groups and ideal class groups. to appear in the Ramanujan Journal (2023)

[20] Kleine, S., Matar, A., Ramdorai, S.: On the $\mathfrak{M}_H(G)$-property. Preprint (2022)

[21] Lang, Serge: Algebra. Graduate Texts in Mathematics, vol. 211, 3rd edn. Springer, New York (2002)

[22] Lim, Meng Fai: Notes on the fine Selmer groups. Asian J. Math. **21**(2), 337–361 (2017)

[23] Lim, M.F.: Structure of fine Selmer groups over $\mathbb{Z}_p$-extensions. preprint (2021)

[24] Mazur, Barry: Rational points of Abelian varieties with values in towers of number fields. Invent. Math. **18**, 183–266 (1972)

[25] Milne, J.S.: Arithmetic Duality Theorems, 2nd edn. BookSurge, LLC, Charleston (2006)

[26] Monsky, Paul: Some invariants of Z_p^d-extensions. Math. Ann. **255**(2), 229–233 (1981)

[27] Mazur, Barry, Rubin, Karl: Finding large Selmer groups. J. Differ. Geom. **70**(1), 1–22 (2005)

[28] Northcott, D.G.: Finite Free Resolutions. Cambridge Tracts in Mathematics, No. 71. Cambridge University Press, Cambridge (1976)

[29] Neukirch, J., Schmidt, A., Wingberg, K.: Cohomology of Number Fields, 2nd edn. Springer, Berlin (2008)

[30] Rohrlich, D.E.: On L-functions of elliptic curves and cyclotomic towers. Invent. Math. **75**, 409–423 (1984)

[31] The Sage Developers. SageMath, the Sage Mathematics Software System (2019)

[32] Washington, Lawrence C.: Introduction to Cyclotomic Fields, volume 83 of Graduate Texts in Mathematics, 2nd edn. Springer, New York (1997)

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