A functional equation for the zeta function of a finitely generated free $\mathbb{Z}_p[G]$-module

Christian Wittmann

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University of the Federal Armed Forces Munich

Department of

COMPUTER SCIENCE

Werner-Heisenberg-Weg 39 • D-85577 Neubiberg

Abstract

Let G be a finite group, p a prime number and $n \in \mathbb{N}$. Following L. Solomon, one can define a zeta-function of the $\mathbb{Z}_p[G]$-module $\mathbb{Z}_p[G]^n$, counting submodules of finite index in $\mathbb{Z}_p[G]^n$. In this article, we present a functional equation for this zeta function. We modify and extend the proof for $n = 1$ in [1], in order to cover our more general situation.

Definitions and the main theorem

If Λ is any (unitary) ring and N a left Λ-module, we define the *zeta function* of N as

$$\zeta_N(s) := \sum_{U \subseteq N} [N : U]^{-s},$$

the sum extending over all Λ-submodules U of finite index in N. If M is another left Λ-module, the *partial zeta function* of N with respect to M is obtained by restricting the sum to those submodules isomorphic to M (over Λ), i.e.

$$\zeta_N(M; s) := \sum_{\substack{U \subseteq N \\ U \cong M}} [N : U]^{-s}.$$

Analogously, if N and M are right Λ-modules, we can define the zeta functions $\zeta_N^{\text{right}}(s)$ and $\zeta_N^{\text{right}}(M; s)$. In either case we consider these functions for those $s \in \mathbb{C}$ such that the sum converges.

Example. Let $\Lambda = \mathbb{Z}$ and n be a positive integer. Then

$$\zeta_{\mathbb{Z}}(s) = \sum_{n \geq 1} n^{-s}$$

is the *Riemann zeta function*, and

$$\zeta_{\mathbb{Z}^n}(s) = \prod_{m=0}^{n-1} \zeta_{\mathbb{Z}}(s - m),$$

cf. [1, §1]. In particular $\zeta_{\mathbb{Z}^n}(s)$ converges for $\mathrm{Re}(s) > n - 1$.

We now fix a finite group G, a prime number p and some positive integer n throughout this paper. Let $R := \mathbb{Z}_p[G]$ be the group ring of G over the ring of p-adic integers. Then R is a $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G]$ and hence is contained in some maximal order $\widetilde{R}$ of $\mathbb{Q}_p[G]$. We put

$$\begin{aligned} \Lambda &:= \mathrm{M}_n(R), \\ \widetilde{\Lambda} &:= \mathrm{M}_n(\widetilde{R}), \\ A &:= \mathrm{M}_n(\mathbb{Q}_p[G]). \end{aligned}$$

Then A is a finite dimensional semisimple $\mathbb{Q}_p$-algebra, $\Lambda \subseteq \widetilde{\Lambda}$ are $\mathbb{Z}_p$-orders in A and $\widetilde{\Lambda}$ is maximal (cf. [2, Th. 26.25]). We define

$$\delta_\Lambda(s) := \frac{\zeta_\Lambda(s)}{\zeta_{\widetilde{\Lambda}}(s)}.$$

Then, according to *Solomon's First Conjecture* proved in [1, Th. 1], $\delta_\Lambda(s)$ is a polynomial in p^{-s} with integer coefficients. We can now state the main result of this paper.

Theorem 1. *The quotient $\delta_\Lambda(s) \in \mathbb{Z}[p^{-s}]$ satisfies the following functional equation:*

$$\delta_\Lambda(s) = [\widetilde{\Lambda} : \Lambda]^{1-2s}\, \delta_\Lambda(1-s).$$

MORITA's Theorem (cf. [3, Sec. 3.12]) induces an isomorphism between the lattice of left ideals of $\Lambda = \mathrm{M}_n(R)$ of finite index in $\mathrm{M}_n(R)$ and the lattice of submodules of finite index in the left R-module R^n. If $I \subseteq \mathrm{M}_n(R)$ and $U \subseteq R^n$ correspond to each other under that isomorphism, the relation

$$[\mathrm{M}_n(R) : I] = [R^n : U]^n$$

is easily verified. Thus we get

$$\zeta_{\mathrm{M}_n(R)}(s) = \zeta_{R^n}(ns),$$

and in the same way

$$\zeta^{\mathrm{right}}_{\mathrm{M}_n(R)}(s) = \zeta^{\mathrm{right}}_{R^n}(ns).$$

But since $\mathrm{M}_n(R)$ has a canonical anti-automorphism, given by

$$(a_{ij}) \mapsto (\varphi(a_{ij}))^T,$$

where

$$\varphi : R \to R, \quad g \mapsto g^{-1} \qquad \text{for all } g \in G$$

is an anti-automorphism of R, the left and right zeta functions coincide in the above situation. Hence

$$\zeta_\Lambda(s) = \zeta_{R^n}(ns) = \zeta^{\mathrm{right}}_{R^n}(ns) = \zeta^{\mathrm{right}}_\Lambda(s).$$

A similar argument shows

$$\zeta_{\widetilde{\Lambda}}(s) = \zeta_{\widetilde{R}^n}(ns) = \zeta^{\mathrm{right}}_{\widetilde{R}^n}(ns) = \zeta^{\mathrm{right}}_{\widetilde{\Lambda}}(s);$$

here the left and right zeta functions are again the same, $\widetilde{R}$ and $\widetilde{\Lambda}$ being maximal orders.

Using these facts we can reformulate the above theorem, leading to a functional equation for the zeta function of the free R-module R^n.

Corollary 2. *Let*

$$\delta_{R^n}(s) := \frac{\zeta_{R^n}(s)}{\zeta_{\widetilde{R}^n}(s)}.$$

Then $\delta_{R^n}(s) \in \mathbb{Z}[p^{-s}]$ satisfies the following functional equation:

$$\delta_{R^n}(s) = [\widetilde{R} : R]^{n^2-2ns}\, \delta_{R^n}(n-s).$$

Lattices on A

Let $t : \mathbb{Q}_p[G] \to \mathbb{Q}_p$ be the $\mathbb{Q}_p$-linear map

$$t\left(\sum_{g\in G} \alpha_g g\right) \mapsto \alpha_1,$$

and define a trace map

$$T : A \to \mathbb{Q}_p, \quad (a_{ij}) \mapsto \sum_{k=1}^{n} t(a_{kk}).$$

Then the pairing $(x, y) \mapsto T(xy)$ is a symmetric, non-degenerate bilinear form, and we can identify A with its linear dual $\mathrm{Hom}_{\mathbb{Q}_p}(A, \mathbb{Q}_p)$ via T. Consider the continuous character

$$\chi : \mathbb{Q}_p \to \mathbb{C}^*, \quad \alpha \mapsto e^{2\pi i\alpha};$$

this is well-defined, if we choose a decomposition $\alpha = \alpha_0 + \alpha_1$ with $\alpha_0 \in \mathbb{Z}[\frac{1}{p}]$, $\alpha_1 \in \mathbb{Z}_p$ and put $e^{2\pi i\alpha} := e^{2\pi i\alpha_0}$. Letting

$$\theta := \chi \circ T \ : \ A \to \mathbb{C}^*,$$

the pairing $(x, y) \mapsto \theta(xy)$ is again symmetric and non-degenerate, and identifies the locally compact abelian group A with its Pontrjagin dual $\widehat{A}$, where

$$\widehat{A} := \mathrm{Hom}^{\mathrm{cont}}(A, S^1)$$

(continuous group homomorphisms), S^1 being the unit circle in $\mathbb{C}$ (cf. [4, Ch. II §5 Th.3]).

Let $M \subseteq A$ be a *left Λ-lattice on A*, i.e. a full $\mathbb{Z}_p$-lattice on A such that $\Lambda M \subseteq M$. We define

$$M^\perp := \{x \in A \mid \forall y \in M: \ \theta(yx) = 1\}.$$

The (algebraic and topological) isomorphism $A \to \widehat{A}$ yields isomorphisms

$$A/M^\perp \cong \widehat{M}$$

and

$$M^\perp \cong \widehat{A/M},$$

so $M^\perp$ is an open subgroup of A by the former and compact by the latter. Hence $M^\perp$ is a full $\mathbb{Z}_p$-lattice on A, and thus obviously a right Λ-lattice on A.

Lemma 3. *Let M be a left Λ-lattice on A.*

a) $M^\perp = \{x \in A \mid Mx \subseteq \Lambda\}$.

b) $M^\perp \cong \mathrm{Hom}_\Lambda(M, \Lambda)$ *as right Λ-modules.*

c) $\Lambda^\perp = \Lambda$.

Proof. a) Since $\theta(\Lambda) = \{1\}$, the inclusion $\supseteq$ follows. Now take $x \in M^\perp$ and $y \in M$. Let $e^{(ij)}$ be the matrix in Λ having a 1 at position (i,j), all other entries being 0. Then $e^{(ij)}y \in M$, and consequently

$$t((yx)_{ij}) = T(e^{(ij)}yx) \in \mathbb{Z}_p$$

for all $i, j = 1, \dots, n$. Replacing $e^{(ij)}$ by $ge^{(ij)}$ for $g \in G$ we get

$$t(g(yx)_{ij}) \in \mathbb{Z}_p \quad (g \in G),$$

and this implies $(yx)_{ij} \in \mathbb{Z}_p[G] = R$, by the definition of t. Thus $yx \in \mathrm{M}_n(R) = \Lambda$.

b) This is clear: Because of $M \otimes \mathbb{Q}_p = \Lambda \otimes \mathbb{Q}_p = A$, every $f \in \mathrm{Hom}_\Lambda(M, \Lambda)$ uniquely extends to a $\widetilde{f} \in \mathrm{Hom}_A(A, A)$. Hence f is given by (right) multiplication with some $x \in A$ satisfying $Mx \subseteq \Lambda$.

c) is a direct consequence of a). □

If $N \subseteq M$ are full $\mathbb{Z}_p$-lattices on A, the index $[M : N]$ is defined and finite. For arbitrary $\mathbb{Z}_p$-lattices M, N on A, we can define a *generalized group index* by

$$(M : N) := \frac{[M : M \cap N]}{[N : M \cap N]}.$$

Lemma 4. *Let M, N be Λ-lattices on A.*

a) $[M : N] = [N^\perp : M^\perp]$ *if* $N \subseteq M$.

b) $(M : N) = (N^\perp : M^\perp)$.

c) $M = M^{\perp\perp}$.

Proof. a) We begin by showing that we can identify

$$M^\perp = \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p), \tag{$*$}$$

where $x \in M^\perp$ corresponds to the map $y \mapsto T(yx)$. Note that $Mx \subseteq \Lambda$ iff $T(Mx) \subseteq \mathbb{Z}_p$. Let $b_1, \ldots, b_r$ be a $\mathbb{Z}_p$-basis of M (with $r = n^2|G|$). Then $b_1, \ldots, b_r$ is a $\mathbb{Q}_p$-basis of A. Let $c_1, \ldots, c_r$ the dual $\mathbb{Q}_p$-basis, i.e. such that $T(b_i c_j) = \delta_{ij}$. Then $c_1, \ldots, c_r$ is a $\mathbb{Z}_p$-basis of $M^\perp$, and $(*)$ follows.

Now choose a $\mathbb{Z}_p$-basis $b_1, \ldots, b_r$ of M such that $\lambda_1 b_1, \ldots, \lambda_r b_r$ is a $\mathbb{Z}_p$-basis of $N \subseteq M$, where $\lambda_i \geq 1$ are integers. Then

$$[M : N] = \lambda_1 \ldots \lambda_r.$$

Let $d_1, \ldots, d_r$ be the dual basis of $N^\perp$ with respect to $\lambda_1 b_1, \ldots, \lambda_r b_r$, i.e.

$$T(\lambda_i b_i d_j) = \delta_{ij}$$

for all i, j. This, however, implies that $\lambda_1 d_1, \ldots, \lambda_r d_r$ is dual to $b_1, \ldots, b_r$, hence is a $\mathbb{Z}_p$-basis of $M^\perp \subseteq N^\perp$. Thus

$$[N^\perp : M^\perp] = \lambda_1 \ldots \lambda_r,$$

and the claim is proved.

b) Using $M^\perp \cap N^\perp = (M + N)^\perp$ we get

$$(N^\perp : M^\perp) = \frac{[N^\perp : (M+N)^\perp]}{[M^\perp : (M+N)^\perp]} \overset{a)}{=} \frac{[M+N : N]}{[M+N : M]} = \frac{[M : M \cap N]}{[N : M \cap N]} = (M : N).$$

c) We have $M \subseteq M^{\perp\perp}$ and

$$(M : M^\perp) = (M^{\perp\perp} : M^\perp) = (M^{\perp\perp} : M)(M : M^\perp),$$

whence $M = M^{\perp\perp}$. □

Fourier transforms and zeta integrals

Let $\mathfrak{S}(A)$ be the space of SCHWARTZ-BRUHAT functions, i.e. locally constant functions $A \to \mathbb{C}$ of compact support (cf. [4, VII §2]). Choose a HAAR measure dx on A. Then for any $\Phi \in \mathfrak{S}(A)$, we define the FOURIER transform $\widehat{\Phi} \in \mathfrak{S}(A)$ by

$$\widehat{\Phi}(y) := \int_A \Phi(x)\theta(xy)dx.$$

We require the measure dx to be *self-dual*, i.e. such that the FOURIER inversion formula

$$\widehat{\widehat{\Phi}}(x) = \Phi(-x)$$

holds for all $\Phi \in \mathfrak{S}(A)$, $x \in A$. Equivalently, $\mu(\Lambda) = 1$ (see Lemma 6 below), where

$$\mu(E) := \int_E dx$$

for any measurable set $E \subseteq A$.

Lemma 5. *Let Φ be the characteristic function of a left Λ-lattice M on A. Then $\widehat{\Phi}$ is the characteristic function of $M^\perp$, multiplied by $\mu(M)$.*

Proof. If $y \in M^\perp$, then $\theta(xy) = 1$ for all $x \in M$. Hence

$$\widehat{\Phi}(y) = \int_M \theta(xy)dx = \int_M dx = \mu(M).$$

On the other hand, if $y \notin M^\perp$, there is an element $x_0 \in M$ with $\theta(x_0 y) \neq 1$, and

$$\begin{aligned}\widehat{\Phi}(y) &= \int_M \theta(xy)dx \\ &= \int_M \theta((x_0 + x)y)dx \\ &= \theta(x_0 y)\int_M \theta(xy)dx \\ &= \theta(x_0 y)\ \widehat{\Phi}(y);\end{aligned}$$

therefore $\widehat{\Phi}(y) = 0$ in this case. □

Lemma 6.

a) $\mu(\Lambda) = 1$.

b) If M is a left Λ-lattice on A, then $\mu(M) = (M : \Lambda)$.

Proof. Let Φ be the characteristic function of $\Lambda = \Lambda^\perp$. Then $\widehat{\Phi}(x) = \mu(\Lambda)\Phi(x)$. Using FOURIER inversion we infer

$$\Phi(x) = \Phi(-x) = \widehat{\widehat{\Phi}}(x) = \mu(\Lambda)^2\Phi(x),$$

whence $\mu(\Lambda) = 1$. This proves a), and b) follows easily since

$$\mu(M) = \frac{\mu(\Lambda)}{(\Lambda : M)} = (M : \Lambda).$$

□

The units A^* form a locally compact topological group (the topology being the subset topology from A). If $x \in A^*$, we let

$$\|x\| := (Nx : N)$$

for any full $\mathbb{Z}_p$-lattice N on A (this is independent of the particular N chosen). Note that $\|xy\| = \|x\|\,\|y\|$ for $x, y \in A^*$. Now

$$d^*x := \frac{dx}{\|x\|}$$

is a (left and right invariant) HAAR measure on A^*. As before, if $E \subseteq A^*$ is a measurable subset, we write

$$\mu^*(E) := \int_E d^*x.$$

Next we define the *zeta integral* for $\Phi \in \mathfrak{S}(A)$ as

$$Z(\Phi; s) := \int_{A^*} \Phi(x)\|x\|^s d^*x.$$

This integral converges certainly for $\mathrm{Re}(s) > 1$, and admits analytic continuation to a meromorphic function on $\mathbb{C}$ (cf. [1, Appendix]).

The main tool towards a proof of Theorem 1 is the following functional equation for the zeta integrals, a special case of which is proven in TATE's thesis.

Theorem 7. *Let $\Phi, \Psi \in \mathfrak{S}(A)$. Then*

$$\frac{Z(\Phi; s)}{Z(\widehat{\Phi}; 1-s)} = \frac{Z(\Psi; s)}{Z(\widehat{\Psi}; 1-s)}.$$

Proof. See [1, Appendix]. Note that a different pairing θ_A (instead of θ) is used there to define the FOURIER transform, and consequently a different Haar measure

$d_A x$ on A is used to obtain self-duality again. But $d_A x = c \cdot dx$ for some constant $c > 0$, and this constant obviously disappears in the above formula. $\square$

If M is a left Λ-lattice on A, we define

$$M^* := \{x \in A^* \mid Mx = M\}.$$

Then M^* is precisely the group of units of the $\mathbb{Z}_p$-order $\{x \in A \mid Mx \subseteq M\}$, hence is a compact open subset of A^* having finite and nonzero measure $\mu^*(M^*)$.

Lemma 8. *Let M be a left Λ-lattice on A, and let Φ be the characteristic function of $M^\perp$. Then*

$$Z(\Phi; s) = \mu^*(M^*)\,(\Lambda : M)^s\,\zeta_\Lambda(M; s).$$

Proof. If $N \subseteq \Lambda$ is a full left ideal with $N \cong M$, there exists $x \in A^*$ with $N = Mx$. Since $Mx \subseteq \Lambda$ we have $x \in A^* \cap M^\perp$, and for arbitrary $x, y \in A^*$:

$$Mx = My \iff xy^{-1} \in M^*.$$

Now the partial zeta function $\zeta_\Lambda(M; s)$ can be rewritten as

$$\begin{aligned} \zeta_\Lambda(M; s) &= \sum_{\overline{y} \in (A^* \cap M^\perp)/M^*} [\Lambda : My]^{-s} \\ &= (\Lambda : M)^{-s} \sum_{\overline{y} \in (A^* \cap M^\perp)/M^*} \|y\|^s. \end{aligned}$$

Further, using Fubini's theorem, we can decompose the zeta integral as

$$Z(\Phi; s) = \sum_{\overline{y} \in A^*/M^*} \int_{M^*} \Phi(yx)\|yx\|^s d^*x.$$

If $y \notin M^\perp$, then $yx \notin M^\perp$ for all $x \in M^*$, hence $\Phi(yx) = 0$ for all $x \in M^*$, while $y \in M^\perp$ leads to

$$\begin{aligned} \int_{M^*} \Phi(yx)\|yx\|^s d^*x &= \|y\|^s \int_{M^*} \|x\|^s d^*x \\ &= \|y\|^s \mu^*(M^*). \end{aligned}$$

Putting everything together we get

$$\begin{aligned} Z(\Phi; s) &= \mu^*(M^*) \sum_{\overline{y} \in (A^* \cap M^\perp)/M^*} \|y\|^s \\ &= \mu^*(M^*)\, (\Lambda : M)^s\, \zeta_\Lambda(M; s), \end{aligned}$$

as desired. $\square$

Lemma 9. *Let Ψ be the characteristic function of the maximal order $\widetilde{\Lambda}$. Then*

$$Z(\Psi; s) = \mu^*(\widetilde{\Lambda}^*)\, \zeta_{\widetilde{\Lambda}}(s).$$

Proof. The proof of the preceding Lemma yields the formula

$$\begin{aligned} Z(\Psi; s) &= \mu^*(\widetilde{\Lambda}^*) \sum_{\overline{y} \in (A^* \cap \widetilde{\Lambda})/\widetilde{\Lambda}^*} \|y\|^s \\ &= \mu^*(\widetilde{\Lambda}^*) \sum_{\overline{y} \in (A^* \cap \widetilde{\Lambda})/\widetilde{\Lambda}^*} [\widetilde{\Lambda} : \widetilde{\Lambda} y]^{-s}. \end{aligned}$$

Since $\widetilde{\Lambda}$ is a maximal order, every left $\widetilde{\Lambda}$-ideal I is isomorphic to $\widetilde{\Lambda}$, i.e. there exists $y \in A^*$ such that $I = \widetilde{\Lambda} y$ (cf. [2, Prop. 31.2]). Therefore the above sum is simply $\zeta_{\widetilde{\Lambda}}(s)$, and the assertion follows. $\square$

Proof of the main theorem

Let Ψ be the characteristic function of $\widetilde{\Lambda}$. By Lemma 5 and Lemma 6 b), $\widehat{\Psi}$ is the characteristic function of $\widetilde{\Lambda}^\perp$, multiplied by the constant factor $[\widetilde{\Lambda} : \Lambda]$. Since $\widetilde{\Lambda}^\perp$ is a full left $\widetilde{\Lambda}$-lattice on A, there exists $\alpha \in A^*$ satisfying $\widetilde{\Lambda}^\perp = \widetilde{\Lambda}\alpha$. Thus

$$\widehat{\Psi}(x) = [\widetilde{\Lambda} : \Lambda]\, \Psi(x\alpha^{-1}).$$

Now

$$\begin{aligned} Z(\widehat{\Psi}; s) &= [\widetilde{\Lambda} : \Lambda] \int_{A^*} \Psi(x\alpha^{-1}) \|x\|^s d^*x \\ &= [\widetilde{\Lambda} : \Lambda] \int_{A^*} \Psi(x) \|\alpha x\|^s d^*x \\ &= [\widetilde{\Lambda} : \Lambda]\, \|\alpha\|^s\, Z(\Psi; s) \\ &= [\widetilde{\Lambda} : \Lambda]\, \|\alpha\|^s\, \mu^*(\widetilde{\Lambda}^*)\, \zeta_{\widetilde{\Lambda}}(s), \end{aligned}$$

where the last equality follows from Lemma 9. Note that

$$\begin{aligned}\|\alpha\|^{-1} &= (\widetilde{\Lambda} : \widetilde{\Lambda}\alpha) \\ &= (\widetilde{\Lambda} : \Lambda)(\Lambda : \widetilde{\Lambda}\alpha) \\ &= (\widetilde{\Lambda} : \Lambda)(\Lambda^{\perp} : \widetilde{\Lambda}^{\perp}) \\ &= [\widetilde{\Lambda} : \Lambda]^2,\end{aligned}$$

whence

$$Z(\widehat{\Psi}; s) = [\widetilde{\Lambda} : \Lambda]^{1-2s}\, \mu^*(\widetilde{\Lambda}^*)\, \zeta_{\widetilde{\Lambda}}(s).$$

Combining this result with Lemma 9 yields the formula

$$\frac{Z(\Psi; s)}{Z(\widehat{\Psi}; 1-s)} = [\widetilde{\Lambda} : \Lambda]^{1-2s}\, \frac{\zeta_{\widetilde{\Lambda}}(s)}{\zeta_{\widetilde{\Lambda}}(1-s)}. \qquad (**)$$

Next let $M \subseteq \Lambda$ be a full left Λ-ideal and let Φ be the characteristic function of $M^{\perp}$. Then $Z(\widehat{\Phi}; s)$ is the characteristic function of $M^{\perp\perp} = M$, multiplied by $\mu(M^{\perp}) = (M^{\perp} : \Lambda) = [\Lambda : M]$. Applying Lemma 8 for both $M^{\perp}$ and M (in the latter case we have to exchange *left* and *right*) we find

$$\begin{aligned}Z(\Phi; s) &= \mu^*(M^*)\, [\Lambda : M]^s\, \zeta_{\Lambda}(M; s), \\ Z(\widehat{\Phi}; s) &= [\Lambda : M]\, \mu^*(M^*)\, (\Lambda : M^{\perp})^s\, \zeta_{\Lambda}^{\mathrm{right}}(M^{\perp}; s) \\ &= [\Lambda : M]^{1-s}\, \mu^*(M^*)\, \zeta_{\Lambda}^{\mathrm{right}}(M^{\perp}; s).\end{aligned}$$

Thus we get the formula

$$\frac{Z(\Phi; s)}{Z(\widehat{\Phi}; 1-s)} = \frac{\zeta_{\Lambda}(M; s)}{\zeta_{\Lambda}^{\mathrm{right}}(M^{\perp}; 1-s)}. \qquad (***)$$

Now the proof of the following Theorem follows from Theorem 7 and the formulas $(**)$, $(***)$.

Theorem 10. *Let $M \subseteq \Lambda$ be a full left Λ-ideal. Then*

$$\frac{\zeta_{\Lambda}(M; s)}{\zeta_{\Lambda}^{\mathrm{right}}(M^{\perp}; 1-s)} = [\widetilde{\Lambda} : \Lambda]^{1-2s}\, \frac{\zeta_{\widetilde{\Lambda}}(s)}{\zeta_{\widetilde{\Lambda}}(1-s)}.$$

Corollary 11.

$$\frac{\zeta_{\Lambda}(s)}{\zeta_{\Lambda}(1-s)} = [\widetilde{\Lambda} : \Lambda]^{1-2s}\, \frac{\zeta_{\widetilde{\Lambda}}(s)}{\zeta_{\widetilde{\Lambda}}(1-s)}.$$

Proof. By the JORDAN-ZASSENHAUS Theorem (cf. [2, §24]), there are only finitely many isomorphism classes of full left Λ-ideals. Let $M_1, \ldots, M_k$ be a set of representatives of these isomorphism classes. Then $M_1^\perp, \ldots, M_k^\perp$ clearly form a set of representatives of the isomorphism classes of full right Λ-ideals. Thus the result follows from the formula in the above Theorem by summing over $M_1, \ldots, M_k$, keeping in mind that $\zeta_\Lambda(s) = \zeta_\Lambda^{\text{right}}(s)$. □

This completes the proof of Theorem 1.

Examples

Suppose that G is a finite group of order m with m coprime to p. Then obviously $|G|$ is invertible in $\mathbb{Z}_p$, whence $R = \widetilde{R}$ is a maximal order (cf. [2, Prop. (27.1)]) and $\Lambda = \widetilde{\Lambda}$. In this case Theorem 1 is trivial since

$$\delta_\Lambda(s) = \delta_{\mathrm{M}_n(\mathbb{Z}_p[G])}(s) = 1.$$

We now consider for the rest of this paper the (more interesting) situation where G is a finite cyclic p-group. Let

$$|G| = p^k,$$

where we fix an integer $k \geq 1$. We first give a formula for the index $[\widetilde{R} : R]$.

Lemma 12. *Let $R = \mathbb{Z}_p[G]$ where G is the cyclic group of order p^k. Let $\widetilde{R} \subseteq \mathbb{Q}_p[G]$ be the maximal order containing R. Then*

$$[\widetilde{R} : R] = p^{1+p+\cdots+p^{k-1}}.$$

Proof. This is proved in [5]. □

Now the functional equation of Corollary 2 reads

$$\delta_{R^n}(s) = p^{(n^2-2ns)(1+p+\cdots+p^{k-1})}\delta_{R^n}(n-s).$$

Substituting

$$x := p^{-s}$$

we can define the polynomial $\widehat{\delta}_{R^n}(x) \in \mathbb{Z}[x]$ by

$$\widehat{\delta}_{R^n}(p^{-s}) = \delta_{R^n}(s).$$

Thus we can reformulate the above equation as follows:

$$\widehat{\delta}_{R^n}(x) = \left(p^{n^2}x^{2n}\right)^{1+p+\cdots+p^{k-1}} \widehat{\delta}_{R^n}\left(\frac{1}{p^n x}\right).$$

We conclude this article by giving formulas for the polynomials $\widehat{\delta}_{R^n}(x)$ in the cases $k = 1$ and $k = 2$. Proofs of these results can be found in [5].

$k = 1$:

Here G is the cyclic group of order p. We have the following formula:

$$\widehat{\delta}_{R^n}(x) = \sum_{e=0}^{n} \left(\begin{bmatrix} n \\ e \end{bmatrix}_p p^{n(n-e)} x^{2(n-e)} \prod_{j=0}^{e-1} \left(1 - p^j x\right) \right),$$

where $\begin{bmatrix} n \\ e \end{bmatrix}_p$ is the number of e-dimensional subspaces of $\mathbb{F}_p^n$, i.e.

$$\begin{bmatrix} n \\ e \end{bmatrix}_p = \frac{(p^n - 1)(p^n - p) \ldots (p^n - p^{e-1})}{(p^e - 1)(p^e - p) \ldots (p^e - p^{e-1})}.$$

The polynomial $\widehat{\delta}_{R^n}(x)$ satisfies the functional equation

$$\widehat{\delta}_{R^n}(x) = p^{n^2} x^{2n} \, \widehat{\delta}_{R^n} \left(\frac{1}{p^n x} \right).$$

$k = 2$:

Here G is the cyclic group of order p^2. In this case we do not know of a nice formula in closed form as before, but there is an algorithm (described in [5]) allowing the computation of the polynomial $\widehat{\delta}_{R^n}(x)$ for arbitrary p and n. We present the results for $p \in \{2, 3, 5\}$ and $n \in \{1, 2, 3\}$.

- $p = 2$ and $n = 1$:

  $8x^6 - 8x^5 + 6x^4 + 3x^2 - 2x + 1,$

- $p = 2$ and $n = 2$:

  $4096x^{12} - 6144x^{11} + 6400x^{10} - 2304x^9 + 2816x^8 - 2304x^7 + 1952x^6$
  $- 576x^5 + 176x^4 - 36x^3 + 25x^2 - 6x + 1,$

- $p = 2$ and $n = 3$:

  $134217728x^{18} - 234881024x^{17} + 278921216x^{16} - 143654912x^{15}$
  $+ 136708096x^{14} - 110100480x^{13} + 102023168x^{12} - 43696128x^{11}$

$+ 17389568x^{10} - 4376576x^9 + 2173696x^8 - 682752x^7 + 199264x^6$
$- 26880x^5 + 4172x^4 - 548x^3 + 133x^2 - 14x + 1,$

- $p = 3$ and $n = 1$:

  $81x^8 - 54x^7 + 36x^6 + 9x^5 + 3x^4 + 3x^3 + 4x^2 - 2x + 1,$

- $p = 3$ and $n = 2$:

  $21x^{16} - 38263752x^{15} + 30823578x^{14} + 708588x^{13} + 1535274x^{12}$
  $+ 3385476x^{11} + 2119203x^{10} - 1355940x^9 + 1167129x^8 - 150660x^7$
  $+ 26163x^6 + 4644x^5 + 234x^4 + 12x^3 + 58x^2 - 8x + 1,$

- $p = 3$ and $n = 3$:

  $150094635296999121x^{24} - 144535574730443598x^{23} + 123122896992600102x^{22}$
  $- 5467553396735679x^{21} + 6377541363277461x^{20} + 1439628076304601 3x^{19}$
  $+ 6581267202259089x^{18} - 4740422864535540x^{17} + 4916296269721980x^{16}$
  $- 868287187367289x^{15} + 200488295095218x^{14} + 15476501507688x^{13}$
  $+ 777744240183x^{12} + 573203759544x^{11} + 275018237442x^{10}$
  $- 44113559283x^9 + 9250878780x^8 - 330368220x^7 + 16987401x^6$
  $+ 1376271x^5 + 22581x^4 - 717x^3 + 598x^2 - 26x + 1,$

- $p = 5$ and $n = 1$:

  $15625x^{12} - 6250x^{11} + 3750x^{10} + 1875x^9 + 125x^8$
  $+ 375x^7 + 25x^6 + 75x^5 + 5x^4 + 15x^3 + 6x^2 - 2x + 1,$

- $p = 5$ and $n = 2$:

  $59604644775390625x^{24} - 28610229492187500x^{23}$
  $+ 18692016601562500x^{22} + 7209777832031250x^{21}$
  $+ 7629394531250x^{20} + 2210998535156250x^{19}$
  $- 56915283203125x^{18} + 450073242187500x^{17}$

$$
\begin{aligned}
&- 12957763671875x^{16} + 90329589843750x^{15} \\
&+ 20233642578125x^{14} - 6640722656250x^{13} \\
&+ 5911884765625x^{12} - 265628906250x^{11} \\
&+ 32373828125x^{10} + 5781093750x^{9} - 33171875x^{8} + 46087500x^{7} \\
&- 233125x^{6} + 362250x^{5} + 50x^{4} + 1890x^{3} + 196x^{2} - 12x + 1,
\end{aligned}
$$

- $p = 5$ and $n = 3$:

$$
\begin{aligned}
&5551115123125782702118158340454101562 5x^{36} \\
&- 27533531010703882202506065368652343750x^{35} \\
&+ 18282226459110737778246402740478515625 0x^{34} \\
&+ 66582117597135948017239570617675781 25x^{33} \\
&- 84406792666413821280002593994140625x^{32} \\
&+ 22291499135462800040841102600097656 25x^{31} \\
&- 97134670795639976859092712402343750x^{30} \\
&+ 45690922888752538710832595825195312 5x^{29} \\
&- 20771263370988890528678894042968750x^{28} \\
&+ 9163908005575649440288543701171875 0x^{27} \\
&+ 1782673319685272872447967529296875 0x^{26} \\
&- 58045001490972936153411865234375 00x^{25} \\
&+ 59192114890553057193756103515625 00x^{24} \\
&- 32667064666282385587692260742187 5x^{23} \\
&+ 48173992687091231346130371093750x^{22} \\
&+ 61979194982349872589111328125 00x^{21} \\
&- 47538805550336837768554687500x^{20} \\
&+ 6599504788994789123535156250 0x^{19} - 560464551913738250732421875x^{18} \\
&+ 5279603831195831298828125 00x^{17} - 30424835552215576171875 00x^{16} \\
&+ 31733347830963134765625 00x^{15} + 19732067404632568359375 0x^{14} \\
&- 1070434374984741210937 5x^{13} + 1551685776586914062500x^{12} \\
&- 121729190966796875 00x^{11} + 2990829534179687 50x^{10} \\
&+ 12299589121093750x^{9} - 223029742187 50x^{8} + 3924820390625x^{7} - 6675043750x^{6} \\
&+ 1225488125x^{5} - 371225x^{4} + 234265x^{3} + 5146x^{2} - 62x + 1.
\end{aligned}
$$

In each case one can verify the functional equation

$$
\widehat{\delta}_{R^n}(x) = \left(p^{n^2} x^{2n}\right)^{1+p} \widehat{\delta}_{R^n}\left(\frac{1}{p^n x}\right)
$$

predicted by Corollary 2.

References

[1] C.J. Bushnell, I. Reiner, *Zeta functions of arithmetic orders and Solomon's Conjectures*, Math. Z. **173** (1980), 135–161.

[2] C.W. Curtis, I. Reiner, *Methods of Representation Theory I*, Wiley-Interscience, 1981.

[3] N. Jacobson, *Basic Algebra II*, Freeman, 1980.

[4] A. Weil, *Basic Number Theory*, Springer, 1974.

[5] C. Wittmann, Dissertation, Universität der Bundeswehr München, to appear.