Relation Algebras are Matrix Algebras over a Suitable Basis

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Abstract

Given a heterogeneous relation algebra $\mathcal{R}$, it is well-known that the algebra of matrices with coefficients from $\mathcal{R}$ is a relation algebra with (not necessarily finite) relational sums. In this paper we want to show that under slightly stronger assumptions the other implication is also true. Every relation algebra $\mathcal{R}$ with relational sums and subobjects is equivalent to an algebra of matrices over a suitable basis. This basis is the full subalgebra $\mathcal{B}$ induced by the integral objects of $\mathcal{R}$. Integral objects may be characterized by their identity morphisms. Furthermore, we show that this representation is not a trivial one since $\mathcal{B}$ is always a proper subalgebra of $\mathcal{R}$. Last but not least, we reprove that every relation algebra may be embedded into a product of simple algebras using our concept of a basis.

1 Introduction

Under certain circumstances, i.e. relational products exist or the point axiom is given, a relation algebra may be represented in the algebra Rel of concrete binary relations between sets. In other words, the algebra may be seen as an algebra of boolean matrices.

As known, not every relation algebra is representable and therefore is not an algebra of boolean matrices. In this paper, we want to show that in every relation algebra $\mathcal{R}$ with relational sums and subobjects it is possible to characterize a full[1] subalgebra $\mathcal{B}$ such that the matrix algebra $\mathcal{B}^+$ with coefficients from $\mathcal{B}$ is equivalent to $\mathcal{R}$. This equivalence is not necessarily an isomorphism since isomorphic objects from $\mathcal{R}$ may be identified under this equivalence. The objects of $\mathcal{B}$ are the integral objects of $\mathcal{R}$. Integral objects are characterized by the fact that their identity morphism is an atom. We call $\mathcal{B}$ the basis of $\mathcal{R}$.

[1] in the sense of category theory

As shown in [5], every relation algebra may be embedded into one with relational sums and subobjects and hence into one which is equivalent to a matrix algebra. This embedding and the equivalence above is not a trivial one. We show that $\mathcal{B}$ is never isomorphic or equivalent to $\mathcal{R}$ and hence, a proper subalgebra of $\mathcal{R}$.

Furthermore, we use our concept to reprove that every relation algebra may be embedded into a product of simple algebras.

The rest of the paper is organized as follows. In Section 2, we briefly recall some basic definitions of the theory of heterogeneous relation algebras. Section 3 is dedicated to matrix algebras with coefficients from a given relation algebra. The integral objects and the basis are introduced in Section 4. Afterwards in Section 5 we prove our main theorem, i.e. a pseudo-representation theorem for heterogeneous relation algebras. Finally, in Section 6 we reprove the theorem mentioned above.

We assume that the reader is familiar with the basic concepts of category theory and the theory of heterogeneous relation algebras. We use the notation of [3].

2 Heterogeneous Relation Algebras

In this section we recall some fundamentals on heterogeneous relation algebras. For further details we refer to [1, 2, 3].

Definition 2.1 *A (heterogeneous abstract) relation algebra is a locally small category $\mathcal{R}$ consisting of a class $\mathrm{Obj}_{\mathcal{R}}$ of objects and a set $\mathcal{R}[A,B]$ of morphisms for all $A,B \in \mathrm{Obj}_{\mathcal{R}}$. The morphisms are usually called relations. Composition is denoted by ";" and identities are denoted by $\mathbb{I}_A \in \mathcal{R}[A,A]$. In addition, there is a totally defined unary operation $\breve{\ }_{AB} : \mathcal{R}[A,B] \longrightarrow \mathcal{R}[B,A]$ between the sets of morphisms, called conversion. The operations satisfy the following rules:*

1. *Every set $\mathcal{R}[A,B]$ carries the structure of a complete atomic boolean algebra with operations $\sqcup_{AB}, \sqcap_{AB}, \overline{}_{AB}$, zero element $\perp\!\!\!\perp_{AB}$, universal element $\top\!\!\!\top_{AB}$, and inclusion ordering $\sqsubseteq_{AB}$.*
2. *The Schröder equivalences*

$$Q;R \sqsubseteq_{AC} S \iff Q\breve{\ };\overline{S} \sqsubseteq_{BC} \overline{R} \iff \overline{S};R\breve{\ } \sqsubseteq_{AB} \overline{Q}$$

hold for relations Q, R and S (where the definedness of one of the three formulae implies that of the other two).

All the indices of elements and operations are usually omitted for brevity and can easily be reinvented. □

In the next lemma we collect some properties we will need throughout this paper. A proof may be found in [1, 2, 3, 4, 5, 6].

Lemma 2.2 *Let $\mathcal{R}$ be a relation algebra, A, B objects of $\mathcal{R}$ and $Q \in \mathcal{R}[A, B]$. Then we have*

1. $Q \sqsubseteq Q; Q^{\smile}; Q$,
2. $\mathbb{T}_{AA}; \mathbb{T}_{AB} = \mathbb{T}_{AB}$,
3. $\mathbb{T}_{AB}; \mathbb{T}_{BB} = \mathbb{T}_{AB}$,
4. $\mathbb{T}_{AB}; \mathbb{T}_{BA}; \mathbb{T}_{AB} = \mathbb{T}_{AB}$. □

An important class of relations are the mappings.

Definition 2.3 *Let $Q \in \mathcal{R}[A, B]$ be a relation.*

1. *Q is called univalent iff $Q^{\smile}; Q \sqsubseteq \mathbb{I}_B$,*
2. *Q is called total iff $\mathbb{I}_A \sqsubseteq Q; Q^{\smile}$ or equivalent $Q; \mathbb{T}_{BA} = \mathbb{T}_{AA}$,*
3. *Q is called a map iff Q is univalent and total.* □

In the next lemma we collect two fundamental facts about univalent relations. A proof may be found in [1, 2, 3, 4, 5, 6].

Lemma 2.4 *Let $Q \in \mathcal{R}[A, B]$ be univalent and $R, S \in \mathcal{R}[B, C]$. Then we have*

1. $Q; (R \sqcap S) = Q; R \sqcap Q; S$,
2. *If Q is further total and hence a mapping then $\overline{Q; R} = Q; \overline{R}$.* □

We define the notion of a homomorphism between relation algebras as usual.

Definition 2.5 *Let $\mathcal{R}$ and $\mathcal{S}$ be relation algebras and $F : \mathcal{R} \to \mathcal{S}$ a functor. Then F is called a homomorphism between relation algebras iff*

1. $F(\bigsqcap_{i \in I} S_i) = \bigsqcap_{i \in I} F(S_i)$,
2. $F(\overline{R}) = \overline{F(R)}$,
3. $F(R^{\smile}) = F(R)^{\smile}$,

hold for all relations R, S_i with $i \in I$. □

A pair of homomorphisms $F : \mathcal{R} \to \mathcal{S}, G : \mathcal{S} \to \mathcal{R}$ is called an equivalence iff $F \circ G$ and $G \circ F$ are naturally isomorphic to the identity functors, e.g. F and G are inverse of each other up to isomorphism.

The relational description of disjoint unions is the relational sum [3, 6]. This construction corresponds to the categorical product[2]. Here we want to generalize this concept to not necessarily finite sets of objects.

Definition 2.6 *Let $\{A_i \mid i \in I\}$ be a set of objects indexed by a set I. An object $\sum\limits_{i\in I} A_i$ together with relations $\iota_j \in \mathcal{R}[A_j, \sum\limits_{i\in I} A_i]$ for all $j \in I$ is called a relational sum of $\{A_i \mid i \in I\}$ iff for all $i, j \in I$ with $i \neq j$ the following holds*

$$\iota_i;\iota_i^{\smile} = \mathbb{I}_{A_i}, \qquad \iota_i;\iota_j^{\smile} = \perp\!\!\!\perp_{A_iA_j}, \qquad \bigsqcup_{i\in I} \iota_i^{\smile};\iota_i = \mathbb{I}_{\sum\limits_{i\in I} A_i}.$$

$\mathcal{R}$ has relational sums iff for every set of objects the relational sum does exist. □

For a set of two objects $\{A, B\}$ this definition corresponds to usual definition of the relational sum. As known categorical products and hence relational sums are unique up to isomorphism.

For given sets of relations $Q_i \in \mathcal{R}[A_i, C]$ and $R_i \in \mathcal{R}[A_i, B_i]$ for all $i \in I$ and relational sums $(\sum\limits_{i\in I} A_i, \iota_i)_{i\in I}$ and $(\sum\limits_{i\in I} B_i, \iota_i)_{i\in I}$ we use the notaion

$$\bigvee_{i\in I} Q_i := \bigsqcup_{i\in I} \iota_i^{\smile};R_i \qquad \sum_{i\in I} R_i := \bigvee_{i\in I} R_i;\iota_i' = \bigsqcup_{i\in I} \iota_i^{\smile};R_i;\iota_i'.$$

$\bigvee\limits_{i\in I} Q_i$ is the biproduct morphism, i.e. it is the unique relation S such that $\iota_i;S = Q_k$ for all $i \in I$.

Lemma 2.7 *Let $\sum\limits_{i\in I} A_i$ be the relational sum of $\{A_i \mid i \in I\}$ and $\sum\limits_{j\in J} B_j$ be the relational sum of $\{B_j \mid j \in J\}$. Then for all $R_{ij} \in \mathcal{R}[A_i, B_j]$ the following holds*

1. $\overline{\bigsqcup\limits_{i\in I, j\in J} \iota_i^{\smile};R_{ij};\iota_j} = \bigsqcup\limits_{i\in I, j\in J} \iota_i^{\smile};\overline{R_{ij}};\iota_j,$
2. $\iota_{k_1}^{\smile};R_{k_1l_1};\iota_{l_1} \sqcap \iota_{k_2}^{\smile};R_{k_2l_2};\iota_{l_2} = \perp\!\!\!\perp_{\sum\limits_{i\in I} A_i \sum\limits_{j\in J} B_j}$ *for all $k_1, k_2 \in I, l_1, l_2 \in J$ with $k_1 \neq k_2$ or $l_1 \neq l_2$.*

Proof:

[2]By conversion, a relation algebra is self-dual. Therefore, a product is also a coproduct and hence a biproduct.

1. By Lemma 2.4 we have

$$\begin{aligned}
\iota_k;\overline{\bigsqcup_{i\in I,j\in J}\iota_i^\smile;R_{ij};\iota_j} &= \overline{\bigsqcup_{i\in I,j\in J}\iota_k;\iota_i^\smile;R_{ij};\iota_j}\\
&= \overline{\bigsqcup_{j\in J}\iota_k;\iota_k^\smile;R_{kj};\iota_j}\\
&= \overline{\bigsqcup_{j\in J}R_{kj};\iota_j}
\end{aligned}$$

for all $k \in I$. By the uniqueness of the biproduct morphism we get

$$\bigsqcup_{i\in I}\iota_i^\smile;\overline{\bigsqcup_{j\in J}R_{ij};\iota_j} = \bigvee_{i\in I}\overline{\bigsqcup_{j\in J}R_{ij};\iota_j} = \overline{\bigsqcup_{i\in I,j\in J}\iota_i^\smile;R_{ij};\iota_j}.$$

Analogously, we conclude from

$$\begin{aligned}
\iota_l;\overline{\bigsqcup_{j\in J}\iota_j^\smile;R_{ij}^\smile} &= \overline{\bigsqcup_{j\in J}\iota_l;\iota_j^\smile;R_{ij}^\smile}\\
&= \overline{\iota_l;\iota_l^\smile;R_{il}^\smile}\\
&= \overline{R_{il}^\smile}\\
&= \overline{R_{il}}^\smile
\end{aligned}$$

by the uniqueness of the biproduct morphism

$$\bigsqcup_{j\in J}\overline{R_{ij}};\iota_j = (\bigsqcup_{j\in J}\iota_j^\smile;\overline{R_{ij}}^\smile)^\smile = (\bigvee_{j\in J}\overline{R_{ij}}^\smile)^\smile = \overline{\bigsqcup_{j\in J}\iota_j^\smile;R_{ij}^\smile}^\smile = \overline{\bigsqcup_{j\in J}R_{ij};\iota_j}$$

and hence

$$\begin{aligned}
\overline{\bigsqcup_{i\in I,j\in J}\iota_i^\smile;R_{ij};\iota_j} &= \bigsqcup_{i\in I}\iota_i^\smile;\overline{\bigsqcup_{j\in J}R_{ij};\iota_j}\\
&= \bigsqcup_{i\in I}\iota_i^\smile;\bigsqcup_{j\in J}\overline{R_{ij}};\iota_j\\
&= \bigsqcup_{i\in I,j\in J}\iota_i^\smile;\overline{R_{ij}};\iota_j.
\end{aligned}$$

2. Suppose $k_1 \neq k_2$. Then by Lemma 2.4 we have

$$\begin{aligned}
\iota_{k_1}^\smile;R_{k_1l_1};\iota_{l_1}\sqcap\iota_{k_2}^\smile;R_{k_2l_2};\iota_{l_2} &= (\bigsqcup_{i\in I}\iota_i^\smile;\iota_i);(\iota_{k_1}^\smile;R_{k_1l_1};\iota_{l_1}\sqcap\iota_{k_2}^\smile;R_{k_2l_2};\iota_{l_2})\\
&= \bigsqcup_{i\in I}\iota_i^\smile;\iota_i;(\iota_{k_1}^\smile;R_{k_1l_1};\iota_{l_1}\sqcap\iota_{k_2}^\smile;R_{k_2l_2};\iota_{l_2})\\
&= \bigsqcup_{i\in I}\iota_i^\smile;(\iota_i;\iota_{k_1}^\smile;R_{k_1l_1};\iota_{l_1}\sqcap\iota_i;\iota_{k_2}^\smile;R_{k_2l_2};\iota_{l_2})\\
&= \bigsqcup_{i\in I}\iota_i^\smile;\perp\!\!\!\perp_{A_i\,\sum\limits_{j\in J}B_j}\\
&= \perp\!\!\!\perp_{\sum\limits_{i\in I}A_i\,\sum\limits_{j\in J}B_j}.
\end{aligned}$$

The case $l_1 \neq l_2$ is shown analogously. □

Subsets may be represented in two different ways inside a relation algebra; by vectors (a relation v such that $v = \mathbb{T}; v$) or partial identities (a relation $l \sqsubseteq \mathbb{I}$). These two concepts are equivalent and may be used to characterize subobjects.

Definition 2.8 *Let $l \in \mathcal{R}[A, A]$ be a partial identity. An object B together with a relation $\psi \in \mathcal{R}[B, A]$ is called subobject of A induced by l iff*

$$\psi; \psi^{\smile} = \mathbb{I}_B, \qquad \psi^{\smile}; \psi = l.$$

A relation algebra has subobjects iff for all partial identities a subobject exists. □

Notice, that we have $Q; R = Q \sqcap R$ for all partial identities Q and R (see [1, 4]).

3 Matrix Algebras

Given a heterogeneous relation algebra $\mathcal{R}$, an algebra of matrices with coefficients from $\mathcal{R}$ may be defined.

Definition 3.1 *Let $\mathcal{R}$ be a relation algebra. The algebra $\mathcal{R}^+$ of matrices with coefficients from $\mathcal{R}$ is defined by:*

1. *The class of objects of $\mathcal{R}^+$ is the collection of all functions from an arbitrary set I to* $\mathrm{Obj}_{\mathcal{R}}$.
2. *For every pair $f : I \to \mathrm{Obj}_{\mathcal{R}}, g : J \to \mathrm{Obj}_{\mathcal{R}}$ of objects from $\mathcal{R}^+$, the set of morphisms $\mathcal{R}^+[f, g]$ is the set of all functions $R : I \times J \to \mathrm{Mor}_{\mathcal{R}}$ such that $R(i, j) \in \mathcal{R}[f(i), g(j)]$ holds.*
3. *For $R \in \mathcal{R}^+[f, g]$ and $S \in \mathcal{R}^+[g, h]$ composition is defined by*
$$(R; S)(i, k) := \bigsqcup_{j \in J} R(i, j); S(j, k).$$
4. *For $R \in \mathcal{R}^+[f, g]$ conversion and negation is defined by*
$$R^{\smile}(j, i) := (R(i, j))^{\smile}, \qquad \overline{R}(i, j) := \overline{R(i, j)}.$$
5. *For $R, S \in \mathcal{R}^+[f, g]$ union and intersection is defined by*
$$(R \sqcup S)(i, j) := R(i, j) \sqcup S(i, j), \qquad (R \sqcap S)(i, j) := R(i, j) \sqcap S(i, j).$$

6. The identity, zero and universal elements are defined by

$$\mathbb{I}_f(i_1, i_2) := \begin{cases} \perp\!\!\!\perp_{f(i_1)f(i_2)} & : i_1 \neq i_2 \\ \mathbb{I}_{f(i_1)} & : i_1 = i_2, \end{cases}$$

$$\perp\!\!\!\perp_{fg}(i,j) := \perp\!\!\!\perp_{f(i)g(j)}, \qquad \top\!\!\!\top_{fg}(i,j) := \top\!\!\!\top_{f(i)g(j)}. \qquad \square$$

Obviously, a morphism in $\mathcal{R}^+$ may be seen as a (in general non-finite) matrix indexed by objects from $\mathcal{R}$. The proof of the following result is an easy exercise and is, therefore, omitted.

Lemma 3.2 $\mathcal{R}^+$ *is a relation algebra.* $\square$

Furthermore, the possibility to build disjoint unions of arbitrary sets indexed by a set gives us the following.

Lemma 3.3 $\mathcal{R}^+$ *has relational sums.* $\square$

Proof: Let $\{f_i : J_i \to \mathrm{Obj}_{\mathcal{R}} \mid i \in I\}$ be a set of objects of $\mathcal{R}^+$. Then the function $h : \sum\limits_{i \in I} J_i \to \mathrm{Obj}_{\mathcal{R}}$ defined by $h(j) := f_i(j)$ iff $j \in J_i$ is also an object of $\mathcal{R}^+$. Now, we define

$$\iota_i(j_1, j_2) := \begin{cases} \perp\!\!\!\perp_{f_i(j_1)h(j_2)} & : j_1 \neq j_2 \\ \mathbb{I}_{f(j_1)} & : j_1 = j_2. \end{cases}$$

An easy verification shows that the above definition gives us the required relational sum. $\square$

4 Integral Objects and the Basis of $\mathcal{R}$

Following the notion used in algebra, we call an object A integral if there are no zero divisors within the subalgebra $\mathcal{R}[A, A]$. Later on, the class of integral objects will define the basis of $\mathcal{R}$.

Definition 4.1 *An object A of a relation algebra is called integral iff $\perp\!\!\!\perp_{AA} \neq \top\!\!\!\top_{AA}$ and for all $Q, R \in \mathcal{R}[A, A]$ the equation $Q; R = \perp\!\!\!\perp_{AA}$ implies either $Q = \perp\!\!\!\perp_{AA}$ or $R = \perp\!\!\!\perp_{AA}$.* $\square$

There are two other simple properties characterizing the integral objects of a relation algebra.

Lemma 4.2 *The following properties are equivalent:*

1. *A is an integral object,*

2. *Every non-zero relation in $\mathcal{R}[A,A]$ is total,*

3. *$\mathbb{I}_A$ is an atom.*

Proof:

1. $\Rightarrow$ 2. : Suppose $\perp\!\!\!\perp_{AA} \neq Q$. From $Q;\top\!\!\!\top_{AA} \sqsubseteq Q;\top\!\!\!\top_{AA}$ we deduce $Q^{\smile};\overline{Q;\top\!\!\!\top_{AA}} = \perp\!\!\!\perp_{AA}$ using the Schröder equivalences. Since A is integral and $Q \neq \perp\!\!\!\perp_{AA}$ we have $Q;\top\!\!\!\top_{AA} = \top\!\!\!\top_{AA}$, i.e. Q is total.

2. $\Rightarrow$ 3. : Suppose $\perp\!\!\!\perp_{AA} \neq Q \sqsubseteq \mathbb{I}_A$. Using that Q is total, we obtain

$$\mathbb{I}_A \sqsubseteq Q;Q^{\smile} \sqsubseteq Q;\mathbb{I}_A^{\smile} = Q.$$

3. $\Rightarrow$ 1. : Suppose $Q;R = \perp\!\!\!\perp_{AA}$. Since $\mathbb{I}_A$ is an atom we have $R;\top\!\!\!\top_{AA} \sqcap \mathbb{I}_A = \perp\!\!\!\perp_{AA}$ or $R;\top\!\!\!\top_{AA} \sqcap \mathbb{I}_A = \mathbb{I}_A$. The first case implies

$$R^{\smile} = R^{\smile};\mathbb{I}_A \sqcap \top\!\!\!\top_{AA} \sqsubseteq R^{\smile};(\mathbb{I}_A \sqcap R;\top\!\!\!\top_{AA}) = R^{\smile};\perp\!\!\!\perp_{AA} = \perp\!\!\!\perp_{AA}$$

and hence $R = \perp\!\!\!\perp_{AA}$. From the second case we conclude

$$Q = Q;\mathbb{I}_A = Q;(R;\top\!\!\!\top_{AA} \sqcap \mathbb{I}_A) \sqsubseteq Q;R;\top\!\!\!\top_{AA} = \perp\!\!\!\perp_{AA};\top\!\!\!\top_{AA} = \perp\!\!\!\perp_{AA}. \qquad \square$$

The special properties of the relations in $\mathcal{R}[A,A]$ mentioned in the last lemma can be transfered to the relation in $\mathcal{R}[A,B]$ for an arbitrary object B.

Lemma 4.3 *Let B be an integral object. Then we have*

1. *if $Q;R = \perp\!\!\!\perp_{AC}$ with $Q \in \mathcal{R}[A,B]$ and $R \in \mathcal{R}[B,C]$ then either $Q = \perp\!\!\!\perp_{AB}$ or $R = \perp\!\!\!\perp_{BC}$,*

2. *if $R \neq \perp\!\!\!\perp_{BC}$ then $R;\top\!\!\!\top_{CD} = \top\!\!\!\top_{BD}$.*

Proof:

1. $Q;R = \perp\!\!\!\perp_{AC}$ implies $Q^{\smile};Q;R;R^{\smile} = Q^{\smile};\perp\!\!\!\perp_{AC};R^{\smile} = \perp\!\!\!\perp_{BB}$. Since B is integral, we have either $Q^{\smile};Q = \perp\!\!\!\perp_{BB}$ or $R;R^{\smile} = \perp\!\!\!\perp_{BB}$. In the first case we conclude using Lemma 2.2 $Q \sqsubseteq Q;Q^{\smile};Q = Q;\perp\!\!\!\perp_{BB} = \perp\!\!\!\perp_{AB}$. The other case is similar.

2. Analogously to 1. $\Rightarrow$ 2. of the last lemma by using 1. $\square$

Notice, that the last lemma implies that all non-zero relations in $\mathcal{R}[B,C]$ are total if B is integral.

Definition 4.4 *Let $\mathcal{R}$ be a relation algebra. The basis $\mathcal{B}_{\mathcal{R}}$ of $\mathcal{R}$ is defined as the full subcategory given by the class of all integral objects.* □

As usual, we omit the index $\mathcal{R}$ in $\mathcal{B}_{\mathcal{R}}$ when its meaning is clear from the context.

Theorem 4.5 *Let $\mathcal{R}$ be a relation algebra with relational sums, and let $\mathcal{B}$ be the basis of $\mathcal{R}$. Then $\mathcal{B}$ is a proper subalgebra of $\mathcal{R}$.*

Proof: Let A be an object of $\mathcal{B}$. We show that the relational sum $A+A$ is not an object of $\mathcal{B}$. Since $\mathcal{R}$ has relational sums, $\mathcal{R}$ and $\mathcal{B}$ are not equivalent. Suppose $\mathbb{I}_{A+A}$ is an atom. Then we have

$$\mathbb{I}_{A+A} = \iota_1^{\smile};\iota_1 \sqcup \iota_2{}^{\smile};\iota_2 \sqsupseteq \iota_1^{\smile};\iota_1.$$

Now, we distinguish two cases:

1. $\iota_1^{\smile};\iota_1 = \perp\!\!\!\perp_{A+AA+A}$: We conclude

$$\begin{aligned} \iota_1 = \iota_1;\iota_1^{\smile};\iota_1 = \iota_1;\perp\!\!\!\perp_{A+AA+A} = \perp\!\!\!\perp_{AA+A} \\ \text{and} \quad \mathbb{I}_A = \iota_1;\iota_1^{\smile} = \perp\!\!\!\perp_{AA+A};\iota_1^{\smile} = \perp\!\!\!\perp_{AA}, \end{aligned}$$

   but the last equality contradicts to $\mathbb{I}_A$ beeing an atom.

2. $\iota_1^{\smile};\iota_1 = \mathbb{I}_{A+A}$: We conclude

$$\begin{aligned} \perp\!\!\!\perp_{AA+A} = \perp\!\!\!\perp_{AA};\iota_1 = \iota_2;\iota_1^{\smile};\iota_1 = \iota_2 \\ \text{and} \quad \mathbb{I}_A = \iota_2;\iota_2^{\smile} = \perp\!\!\!\perp_{AA+A};\iota_2^{\smile} = \perp\!\!\!\perp_{AA}. \end{aligned}$$

   As in case 1 this is a contradiction. □

The last lemma has shown that the definition of the basis of an algebra is not senseless, i.e. the basis usually does not correspond to the whole algebra.

In the rest of this section we want to define an equivalence relation $\approx$ on the basis of $\mathcal{R}$. Later on, it turns out that the equivalence classes of $\approx$ characterize the simple components of the algebra.

Lemma 4.6 *Let A be an integral object. Then we have $\top\!\!\!\top_{AB};\top\!\!\!\top_{BA} \in \{\perp\!\!\!\perp_{AA}, \top\!\!\!\top_{AA}\}$.*

Proof: Since $\mathbb{I}_A$ is an atom $\top\!\!\!\top_{AB};\top\!\!\!\top_{BA} \sqcap \mathbb{I}_A$ is either $\perp\!\!\!\perp_{AA}$ or $\mathbb{I}_A$. Suppose $\top\!\!\!\top_{AB};\top\!\!\!\top_{BA} \sqcap \mathbb{I}_A = \perp\!\!\!\perp_{AA}$. Then we have

$$\begin{aligned} \top\!\!\!\top_{BA} &= \top\!\!\!\top_{BA};\mathbb{I}_A \sqcap \top\!\!\!\top_{BA} \\ &\sqsubseteq \top\!\!\!\top_{BA};(\mathbb{I}_A \sqcap \top\!\!\!\top_{AB};\top\!\!\!\top_{BA}) \\ &= \top\!\!\!\top_{BA};\perp\!\!\!\perp_{AA} \\ &= \perp\!\!\!\perp_{BA}. \end{aligned}$$

It follows $\mathbb{T}_{AB};\mathbb{T}_{BA} = \mathbb{T}_{AB};\perp\!\!\!\perp_{BA} = \perp\!\!\!\perp_{AA}$. If $\mathbb{T}_{AB};\mathbb{T}_{BA} \sqcap \mathbb{I}_A = \mathbb{I}_{AA}$ we conclude using Lemma 2.2

$$\mathbb{T}_{AA} = \mathbb{T}_{AA};\mathbb{I}_A \sqsubseteq \mathbb{T}_{AA};\mathbb{T}_{AB};\mathbb{T}_{BA} = \mathbb{T}_{AB};\mathbb{T}_{BA}. \qquad \square$$

The last lemma leads to the following definition.

Definition 4.7 $A \approx B \quad :\Longleftrightarrow \quad \mathbb{T}_{AB};\mathbb{T}_{BA} = \mathbb{T}_{AA}$. $\square$

To show that $\approx$ is an equivalence relation on the class of integral objects we need the following lemma.

Lemma 4.8 *Let A and B be integral objects. Then we have*

1. $\mathbb{T}_{AB};\mathbb{T}_{BA} = \mathbb{T}_{AA}$ *if and only if* $\mathbb{T}_{BA};\mathbb{T}_{AB} = \mathbb{T}_{BB}$,
2. $\mathbb{T}_{AB};\mathbb{T}_{BA} = \perp\!\!\!\perp_{AA}$ *if and only if* $\mathbb{T}_{BA};\mathbb{T}_{AB} = \perp\!\!\!\perp_{BB}$.

Proof:

1. Suppose $\mathbb{T}_{AB};\mathbb{T}_{BA} = \mathbb{T}_{AA}$ and $\mathbb{T}_{BA};\mathbb{T}_{AB} = \perp\!\!\!\perp_{BB}$. Then using Lemma 2.2

   $$\mathbb{T}_{AB} = \mathbb{T}_{AB};\mathbb{T}_{BA};\mathbb{T}_{AB} = \mathbb{T}_{AB};\perp\!\!\!\perp_{BB} = \perp\!\!\!\perp_{AB}$$

   we get a contradiction. The other implication follows by duality.
2. The assertion follows from 1. and Lemma 4.6. $\square$

The last lemma and Lemma 4.6 show that objects $A \not\approx B$ are characterized by the equation $\mathbb{T}_{AB};\mathbb{T}_{BA} = \perp\!\!\!\perp_{AA}$.

Lemma 4.9 $\approx$ *is an equivalence relation on the basis of $\mathcal{R}$.*

Proof: By Lemma 2.2 $\approx$ is reflexive. Symmetry is implied by Lemma 4.8. Suppose $A \approx B$ and $B \approx C$. By definition we have $\mathbb{T}_{AB};\mathbb{T}_{BA} = \mathbb{T}_{AA}$ and $\mathbb{T}_{BC};\mathbb{T}_{CB} = \mathbb{T}_{BB}$. By Lemma 4.8 we get $\mathbb{T}_{BA};\mathbb{T}_{AB} = \mathbb{T}_{BB}$ and $\mathbb{T}_{CB};\mathbb{T}_{BC} = \mathbb{T}_{CC}$. Using Lemma 2.2 we conclude

$$\begin{aligned}
\mathbb{T}_{CC} &= \mathbb{T}_{CB};\mathbb{T}_{BC} \\
&= \mathbb{T}_{CB};\mathbb{T}_{BB};\mathbb{T}_{BC} \\
&= \mathbb{T}_{CB};\mathbb{T}_{BA};\mathbb{T}_{AB};\mathbb{T}_{BC} \\
&\sqsubseteq \mathbb{T}_{CA};\mathbb{T}_{AC}
\end{aligned}$$

and hence $A \approx C$. $\square$

The equivalence classes of $\approx$ are in a way indepentend.

Lemma 4.10 *Let A and B integral objects. Then the following properties are equivalent:*

1. $A \not\approx B$,
2. $\mathbb{T}_{AB} = \perp\!\!\!\perp_{AB}$.

Proof:

1. $\Rightarrow$ 2. : Since $A \not\approx B$ we have $\mathbb{T}_{AB};\mathbb{T}_{BA} = \perp\!\!\!\perp_{AA}$ by Lemma 4.6. From this we conclude $\mathbb{T}_{AB} = \perp\!\!\!\perp_{AB}$ because otherwise Lemma 4.3 would imply $\mathbb{T}_{AB};\mathbb{T}_{BA} = \mathbb{T}_{AA}$.

2. $\Rightarrow$ 1. : We immediately conclude

$$\mathbb{T}_{AB};\mathbb{T}_{BA} = \perp\!\!\!\perp_{AB};\mathbb{T}_{BA} = \perp\!\!\!\perp_{AA} \neq \mathbb{T}_{AA}. \qquad \Box$$

Notice, that the last Lemma implies that $R = \perp\!\!\!\perp_{AB}$ for all $R \in \mathcal{R}[A,B]$ if $A \not\approx B$.

5 A Pseudo Representation Theorem

Now, we are able to prove our main theorem.

Theorem 5.1 *Let $\mathcal{R}$ be a relation algebra with relational sums and subobjects and $\mathcal{B}$ the basis of $\mathcal{R}$. Then $\mathcal{R}$ and $\mathcal{B}^+$ are equivalent.*

Proof: First, we show that every object A of $\mathcal{R}$ is isomorphic to a relational sum $\sum\limits_{i\in I} A_i$ of objects from $\mathcal{B}$. Let $\{l_i \mid i \in I\}$ be set of all atoms $l_i \sqsubseteq \mathbb{I}_A$. Because $\mathcal{R}$ has subobjects, this gives us a set $\{A_i \mid i \in I\}$ of objects and a set $\{\psi_i \mid i \in I\}$ of morphisms with

$$\psi_i;\psi_i^\smile = \mathbb{I}_{A_i}, \qquad \psi_i^\smile;\psi_i = l_i.$$

Together with the computations

$$\begin{aligned} &\psi_i^\smile;\psi_i;\psi_j^\smile;\psi_j = l_i;l_j = l_i \sqcap l_j = \perp\!\!\!\perp_{AA},\\ &\psi_i;\psi_j^\smile = \psi_i;\psi_i^\smile;\psi_i;\psi_j^\smile;\psi_j;\psi_j^\smile = \psi_i;\perp\!\!\!\perp_{AA};\psi_j^\smile = \perp\!\!\!\perp_{A_iA_j}\\ \text{and}\quad &\bigsqcup_{i\in I}\psi_i^\smile;\psi_i = \bigsqcup_{i\in I} l_i = \mathbb{I}_A. \end{aligned}$$

and the uniqueness of a relational sum, we have $A \cong \sum\limits_{i\in I} A_i$.

Suppose $R \sqsubseteq \mathbb{I}_{A_i}$. Then we have

$$\psi_i^\smile;R;\psi_i \sqsubseteq \psi_i^\smile;\psi_i = l_i.$$

Now, we distinguish two cases:

1. $\psi_i^{\smile}; R; \psi_i = \perp\!\!\!\perp_{AA}$: We conclude

$$R = \psi_i; \psi_i^{\smile}; R; \psi_i; \psi_i^{\smile} = \psi_i; \perp\!\!\!\perp_{AA}; \psi_i^{\smile} = \perp\!\!\!\perp_{A_iA_i}.$$

2. $\psi_i^{\smile}; R; \psi_i = l_i$: We conclude

$$R = \psi_i; \psi_i^{\smile}; R; \psi_i; \psi_i^{\smile} = \psi_i; l_i; \psi_i^{\smile} = \psi_i; \psi_i^{\smile}; \psi_i; \psi_i^{\smile} = \mathbb{I}_{A_i}.$$

This shows that $\mathbb{I}_{A_i}$ is an atom and hence A_i in $\mathcal{B}$. Now, we define the required equivalence $F : \mathcal{R} \to \mathcal{B}^+, G : \mathcal{B}^+ \to \mathcal{R}$ by

$$\begin{aligned}
F(A) &:= f : I \to \mathrm{Obj}_{\mathcal{B}} \text{ with } f(i) = A_i,\\
F(R) &:= h : I_1 \times I_2 \to \mathrm{Mor}_{\mathcal{B}} \text{ with } h(i_1, i_2) = \psi_{i_1}; R; \psi_{i_2}^{\smile},\\
G(f) &:= \sum_{i\in I} f(i),\\
G(h) &:= \bigsqcup_{i\in I, j\in J} \psi_i^{\smile}; h(i,j); \psi_j
\end{aligned}$$

for all $R \in \mathcal{R}[A,B]$, objects $A \cong \sum_{i\in I_1} A_i, B \cong \sum_{i\in I_2} B_i$, $f \in \mathrm{Obj}_{\mathcal{B}^+}$ and $h \in \mathcal{B}^+[f,g]$. By Lemma 2.4 and the computations

$$\begin{aligned}
F(\mathbb{I}_A)(i_1,i_2) &= \psi_{i_1}; \mathbb{I}_A; \psi_{i_2}^{\smile}\\
&= \psi_{i_1}; \psi_{i_2}^{\smile}\\
&= \begin{cases} \mathbb{I}_{A_{i_1}} & : \ i_1 = i_2 \\ \perp\!\!\!\perp_{A_{i_1}A_{i_2}} & : \ i_1 \neq i_2 \end{cases}\\
&= \mathbb{I}_f(i_1,i_2),\\
(F(Q); F(R))(i,k) &= \bigsqcup_{j\in J} F(Q)(i,j); F(R)(j,k)\\
&= \bigsqcup_{j\in J} \psi_i; Q; \psi_j^{\smile}; \psi_j; R; \psi_k^{\smile}\\
&= \psi_i; Q; (\bigsqcup_{j\in J} \psi_j^{\smile}; \psi_j); R; \psi_k^{\smile}\\
&= \psi_i; Q; R; \psi_k^{\smile}\\
&= F(Q;R)(i,k),\\
(F(Q) \sqcap F(S))(i,j) &= F(Q)(i,j) \sqcap F(S)(i,j)\\
&= \psi_i; Q; \psi_j^{\smile} \sqcap \psi_i; S; \psi_j^{\smile}\\
&= \psi_i; (Q \sqcap S); \psi_j^{\smile}\\
&= F(Q \sqcap S)(i,j),\\
\overline{F(Q)}(i,j) &= \overline{F(Q)(i,j)}\\
&= \overline{\psi_i; Q; \psi_j^{\smile}}\\
&= \psi_i; \overline{Q}; \psi_j^{\smile}
\end{aligned}$$

$$
\begin{aligned}
&= F(\overline{Q}),\\
(F(Q)^{\smile})(j,i) &= (F(Q)(i,j))^{\smile}\\
&= (\psi_i; Q; \psi_j^{\smile})^{\smile}\\
&= \psi_j; Q^{\smile}; \psi_i^{\smile}\\
&= F(Q^{\smile})(j,i)
\end{aligned}
$$

is F a homomorphism. We have shown $F(Q) \sqcap F(S) = F(Q \sqcap S)$. The more general case is proven analogously. Conversely, using Lemma 2.4 and 2.7 we get

$$
\begin{aligned}
G(\mathbb{I}_f) &= \bigsqcup_{i_1,i_2 \in I} \psi_{i_1}^{\smile}; \mathbb{I}_f(i_1,i_2); \psi_{i_2}\\
&= \bigsqcup_{i \in I} \psi_i^{\smile}; \mathbb{I}_{f(i)}; \psi_i\\
&= \bigsqcup_{i \in I} \psi_i^{\smile}; \psi_i\\
&= \bigsqcup_{i \in I} l_i\\
&= \mathbb{I}_A,\\
G(f); G(g) &= (\bigsqcup_{i \in I, j \in J} \psi_i^{\smile}; f(i,j); \psi_j); (\bigsqcup_{j \in J, k \in K} \psi_j^{\smile}; g(j,k); \psi_k)\\
&= \bigsqcup_{i \in I, j_1, j_2 \in J, k \in K} \psi_i^{\smile}; f(i,j_1); \psi_{j_1}; \psi_{j_2}^{\smile}; g(j_2,k); \psi_k\\
&= \bigsqcup_{i \in I, j \in J, k \in K} \psi_i^{\smile}; f(i,j); \psi_j; \psi_j^{\smile}; g(j,k); \psi_k\\
&= \bigsqcup_{i \in I, k \in K} \psi_i^{\smile}; (\bigsqcup_{j \in J} f(i,j); g(j,k)); \psi_k\\
&= \bigsqcup_{i \in I, k \in K} \psi_i^{\smile}; (f;g)(i,k); \psi_k\\
&= G(f;g),\\
G(f) \sqcap G(h) &= \bigsqcup_{i \in I, j \in J} \psi_i^{\smile}; f(i,j); \psi_j \sqcap \bigsqcup_{i \in I, j \in J} \psi_i^{\smile}; h(i,j); \psi_j\\
&= \bigsqcup_{i_1, i_2 \in I, j_1, j_2 \in J} \psi_{i_1}^{\smile}; f(i_1,j_1); \psi_{j_1} \sqcap \psi_{i_2}^{\smile}; h(i_2,j_2); \psi_{j_2}\\
&= \bigsqcup_{i \in I, j \in J} \psi_i^{\smile}; f(i,j); \psi_j \sqcap \psi_i^{\smile}; h(i,j); \psi_j\\
&= \bigsqcup_{i \in I, j \in J} \psi_i^{\smile}; (f(i,j) \sqcap h(i,j)); \psi_j\\
&= \bigsqcup_{i \in I, j \in J} \psi_i^{\smile}; (f \sqcap h)(i,j); \psi_j\\
&= G(f \sqcap h),
\end{aligned}
$$

$$\begin{aligned}
\overline{G(f)} &= \overline{\bigsqcup_{i\in I, j\in J} \psi_i^{\smile}; f(i,j); \psi_j} \\
&= \bigsqcup_{i\in I, j\in J} \psi_i^{\smile}; \overline{f(i,j)}; \psi_j \\
&= \bigsqcup_{i\in I, j\in J} \psi_i^{\smile}; \overline{f}(i,j); \psi_j \\
&= G(\overline{f}), \\
G(f)^{\smile} &= (\bigsqcup_{i\in I, j\in J} \psi_i^{\smile}; f(i,j); \psi_j)^{\smile} \\
&= \bigsqcup_{i\in I, j\in J} \psi_j^{\smile}; f(i,j)^{\smile}; \psi_i \\
&= \bigsqcup_{i\in I, j\in J} \psi_j^{\smile}; f^{\smile}(j,i); \psi_i \\
&= G(f^{\smile}).
\end{aligned}$$

Moreover, we have $(G\circ F)(A) = \sum_{i\in I} A_i \cong A$ such that there is a natural isomorphism between $G \circ F$ and the identity on $\mathcal{R}$. Conversely, we have

$$\begin{aligned}
(F\circ G)(f)(i,j) &= F(\bigsqcup_{i\in I, j\in J} \psi_i^{\smile}; f(i,j); \psi_j) \\
&= \psi_i; (\bigsqcup_{i\in I, j\in J} \psi_i^{\smile}; f(i,j); \psi_j); \psi_j^{\smile} \\
&= f(i,j).
\end{aligned}$$

□

In [5] it had been shown that every relation algebra may be embedded into one with relational sums and subobjects. Together, we gain the following corollary.

Corollary 5.2 *Every heterogeneous relation algebra may be embedded into an algebra which is equivalent to a matrix algebra over a suitable basis.* □

6 Simple Relation Algebras

It is known that every homogeneous relation algebra[3] may be embedded into a product of simple algebras. This theorem is an application of general concept from universal algebra. In [5] it was shown that this theorem can be extended to arbitrary heterogeneous relation algebras.

In this section, we want to reprove this theorem using our notion of the basis of a relation algebra and the induced equivalence relation $\approx$.

[3] a relation algebra with just one object

To avoid any set-theoretic problems we require that all relation algebras in this section be small, i.e. the collections of all morphisms and of all objects should be sets.

Definition 6.1 *A collection $\equiv$ of equivalence relations $\equiv_{AB}$ on $\mathcal{R}[A,B]$ is called a congruence iff*

1. $Q^{\smile} \equiv_{BA} R^{\smile}$ *and* $\overline{Q} \equiv_{AB} \overline{R}$ *for all* Q, R *with* $Q \equiv_{AB} R$,
2. $Q;S \equiv_{AC} R;T$ *for all* Q, R *and* S, T *with* $Q \equiv_{AB} R$ *and* $S \equiv_{BC} T$,
3. $Q \equiv_{AB} \bigsqcap_{k \in K} R_k$ *for all* Q, R_k *with* $Q \equiv_{AB} R_k$ *for all* $k \in K$. □

As in universal algebra we define the concept of simple relation algebras.

Definition 6.2 *A relation algebra is called simple iff there at most two congruences.*

It is possible to characterize simple algebras by just one equation, the so-called Tarski-rule.

Lemma 6.3 *Let $\mathcal{R}$ be a relation algebra. Then the following properties are equivalent*

1. $\mathcal{R}$ *is simple,*
2. $Q \neq \perp\!\!\!\perp_{AB}$ *implies* $\top\!\!\!\top_{CA}; Q; \top\!\!\!\top_{BD} = \top\!\!\!\top_{CD}$. □

A proof can be found in [5].

Lemma 6.4 *Let $\mathcal{R}$ be a relation algebra with relational sums and subobjects such that all objects of basis $\mathcal{B}$ are equivalent (in resp. to $\approx$). Then $\mathcal{R}$ is simple.*

Proof: We show that $\mathcal{B}^+$ is simple. The equivalence of $\mathcal{B}^+$ and $\mathcal{R}$ then implies the assertion. Let be $e: I \to \mathcal{B}, f: J \to \mathcal{B}, g: K \to \mathcal{B}$ and $h: L \to \mathcal{B}$ objects of $\mathcal{B}^+$ and $\perp\!\!\!\perp_{fg} \neq R \in \mathcal{B}^+[f,g]$. By definition there is a $j' \in J$ and a $k' \in K$ such that $R(j',k') \neq \perp\!\!\!\perp_{f(j')g(k')}$. From Lemma 4.3 and the fact that all objects of $\mathcal{B}$ are equivalent we conclude

$$\top\!\!\!\top_{e(i)f(j')}; R(j',k'); \top\!\!\!\top_{g(k')h(l)} = \top\!\!\!\top_{e(i)f(j')}; \top\!\!\!\top_{f(j')h(l)} = \top\!\!\!\top_{e(i)h(l)}$$

for all $i \in I$ and $l \in L$. This gives us

$$\begin{aligned}
(\top\!\!\!\top_{ef}; R; \top\!\!\!\top_{gh})(i,l) &= \bigsqcup_{j \in J, k \in K} (\top\!\!\!\top_{ef}(i,j); R(j,k); \top\!\!\!\top_{gh}(k,l)) \\
&= \bigsqcup_{j \in J, k \in K} (\top\!\!\!\top_{e(i)f(j)}; R(j,k); \top\!\!\!\top_{g(k)h(l)}) \\
&= \top\!\!\!\top_{e(i)h(l)} \\
&= \top\!\!\!\top_{eh}(i,l)
\end{aligned}$$

and hence $\mathbb{T}_{ef}; R; \mathbb{T}_{gh} = \mathbb{T}_{eh}$. □

Let $B_{\approx}$ the set of equivalence classes of $\approx$, and $\mathcal{B}_k$ be the full subcategory of $\mathcal{B}$ induced by the equivalence class k. By the last lemma $\mathcal{B}_k^+$ is simple.

Definition 6.5 *Let K be a set, and $\mathcal{R}_k$ for all $k \in K$ be relation algebras. The product relation algebra $\prod\limits_{k\in K} \mathcal{R}_k$ is defined as follows:*

1. *An object of $\prod\limits_{k\in K} \mathcal{R}_k$ is a function $f : K \to \bigcup\limits_{k\in K} \mathrm{Obj}_{\mathcal{R}_k}$ such that $f(k) \in \mathrm{Obj}_{\mathcal{R}_k}$.*

2. *A morphism in $\prod\limits_{k\in K} \mathcal{R}_k[A,B]$ is a function $Q : K \to \bigcup\limits_{k\in K} \mathcal{R}_k[A,B]$ such that $Q(k) \in \mathcal{R}_k[A,B]$.*

3. *The operations and constants are defined in componentwise manner by*

$$\begin{aligned}
(Q;S)(k) &:= Q(k);S(k),\\
(Q \sqcap R)(k) &:= Q(k) \sqcap R(k),\\
(Q \sqcup R)(k) &:= Q(k) \sqcup R(k),\\
Q^{\smile}(k) &:= Q(k)^{\smile},\\
\overline{Q}(k) &:= \overline{Q(k)},\\
\mathbb{I}_f(k) &:= \mathbb{I}_{f(k)},\\
\mathbb{T}_{fg}(k) &:= \mathbb{T}_{f(k)g(k)},\\
\perp\!\!\!\perp_{fg}(k) &:= \perp\!\!\!\perp_{f(k)g(k)},
\end{aligned}$$

for all $Q, R \in \prod\limits_{k\in K} \mathcal{R}_k[f,g]$ and $S \in \prod\limits_{k\in K} \mathcal{R}_k[g,h]$. □

An easy verification shows that $\prod\limits_{k\in K} \mathcal{R}_k$ is indeed a relation algebra.

Theorem 6.6 *Let $\mathcal{R}$ be a small relation algebra. Then $\mathcal{B}^+$ and $\prod\limits_{k\in B_{\approx}} \mathcal{B}_k^+$ are isomorphic.*

Proof: Let $f : I \to \mathrm{Obj}_{\mathcal{B}}$ and $g : J \to \mathrm{Obj}_{\mathcal{B}}$ be objects of $\mathcal{B}^+$ and $R \in \mathcal{B}^+[f,g]$. Furthermore, let

$I_k := \{i \in I \mid f(i) \text{ is a object of } \mathcal{B}_k\}$,
$J_k := \{j \in J \mid g(j) \text{ is a object of } \mathcal{B}_k\}$,
$f_k : I_k \to \mathrm{Obj}_{\mathcal{B}_k}$ such that $f_k(i) = f(i)$,
$R_k : I_k \times J_k \to \mathrm{Mor}_{\mathcal{B}_k}$ such that $R_k(i,j) = R(i,j)$.

Then we define a functor $F : \mathcal{B}^+ \to \prod_{k \in B_\approx} \mathcal{B}_k^+$ by

$$\begin{array}{rcl} F(f)(k) & := & f_k, \\ F(R)(k) & := & R_k. \end{array}$$

Using Lemma 4.10 we get

$$\begin{array}{rcl} F(R;S)(k)(i,l) & = & (R;S)_k(i,l) \\ & = & (R;S)(i,l) \\ & = & \bigsqcup\limits_{j \in J} R(i,j);S(j,l) \\ & = & \bigsqcup\limits_{j \in J_k} R(i,j);S(j,l) \\ & = & \bigsqcup\limits_{j \in J_k} R_k(i,j);S_k(j,l) \\ & = & \bigsqcup\limits_{j \in J_k} F(R)(k)(i,j);F(S)(k)(j,l) \\ & = & (F(R)(k);F(S)(k))(i,l) \end{array}$$

and hence $F(R;S) = F(R);F(S)$. An easy verification shows the other required properties of F and is, therefore, omitted. □

Combining our two main theorems we aim the following.

Corollary 6.7 *Let $\mathcal{R}$ be a small relation algebra with relational sums and subobjects. Then $\mathcal{R}$ and $\prod_{k \in B_\approx} \mathcal{B}_k^+$ are equivalent.* □

Corollary 6.8 *Every small heterogeneous relation algebra may be embedded into an algbra which is equivalent to a product of simple matrix algebras.* □

7 Conclusion

In this paper we have shown that every relation algebra $\mathcal{R}$ may be considered as a subalgebra of a matrix algebra over a suitable basis. This basis is a proper subalgebra of the global completion [1] of $\mathcal{R}$. This shows that a lot of non-finite algebras are completely determined by a finite subalgebra.

The computer system RELVIEW works with Boolean matrices and hence with concrete relations to visualize computations with them. Using the result of this paper, it seems possible to build another computer system working with arbitrary heterogeneous relation algebras. These algebras may be represented by a matrix algebra over a basis given by the user.

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