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Brezzi–Douglas–Marini interpolation on anisotropic simplices and prisms

Numerical analysis / *Analyse numérique*

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Interpolation Brezzi–Douglas–Marini sur les simplexes et prismes anisotropes

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Abstract. The Brezzi–Douglas–Marini interpolation error on anisotropic elements has been analyzed in two recent publications, the first focusing on simplices with estimates in L^2, the other considering parallelotopes with estimates in terms of L^p-norms. This contribution provides generalized estimates for anisotropic simplices for the L^p case, $1 \le p \le \infty$, and shows new estimates for anisotropic prisms with triangular base.

Résumé. L'erreur d'interpolation de Brezzi–Douglas–Marini sur les éléments anisotropes a été analysée dans deux publications récentes, la première se concentrant sur les simplices avec des estimations dans L^2, l'autre considérant les parallelotopes avec des estimations en termes de normes L^p. Notre contribution fournit des estimations généralisées pour les simplexes anisotropes pour le cas L^p, $1 \le p \le \infty$, et montre de nouvelles estimations pour les prismes anisotropes à base triangulaire.

1. Introduction

The Brezzi–Douglas–Marini (BDM) finite element [8] was introduced to approximate $\boldsymbol{H}_{\mathrm{div}}$ by polynomials. This proves useful for problems of incompressible fluid flow, where recent approaches employ $\boldsymbol{H}_{\mathrm{div}}$-conforming discretizations to approximate the velocity solution. The corresponding interpolation operator can also be used as reconstruction operator to gain pressure-robust methods in the spirit of [7]. Boundary layers or edge singularities in these problems require the use of anisotropic, i.e., highly stretched elements, so interpolation error estimates for such settings are required, see [3]. In [2], these estimates were shown for simplicial elements in

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terms of the L^2-norm, and [5] contains estimates for anisotropic parallelotopes in terms of L^p-norms.

The main focus in [5] is on estimates in the H_{div}-norm and while the technique for these proofs can also be used for simplices, it is not applicable for the case of prisms as defined in [8], as here the commuting diagram property [5, (3)] is not satisfied. Interestingly, it is neither satisfied on cubes as defined in [8]; here [5] uses the original definition of the BDM elements, see, e.g., [4].

In the following sections we provide a generalization of the results from [2] to L^p spaces, $1 \le p \le \infty$, and show new interpolation error estimates for anisotropic triangular prisms. The results with detailed proofs are contained in the author's PhD thesis [6] that is to appear.

Notation

Vectors, vector valued functions and the spaces of such functions are set in bold and we use $I_n = \{1, \dots, n\}$ for index sets. The spatial dimension is denoted by d and the expression $a \lesssim b$ means there is a positive constant C so that $a \le Cb$. The norms of the Sobolev spaces $W^{m,p}(G)$ are denoted by $\|\cdot\|_{m,p,G}$, multi-indices by $\boldsymbol{\alpha}$. The directional derivative in direction $\boldsymbol{l}$ is denoted by $\frac{\partial}{\partial \boldsymbol{l}}$.

2. Error estimates on anisotropic simplices

We state the anisotropic interpolation error estimates on simplices in terms of L^p-norms, which is a generalization of the results from [2], where only $p = 2$ was considered. As the proofs are largely analogous to those in [2], we omit them here. The principal idea of the proof is to first show stability estimates on the reference elements and then transfer them in two steps first to an element of a reference family using an affine transformation with a diagonal matrix and then to the general element.

Recall that a simplex is said to satisfy the *maximum angle condition* if all angles within and between facets are bounded by a constant $\bar{\phi} < \pi$. In addition, it satisfies the *regular vertex property* if for one vertex there is a constant $\bar{c} > 0$ so that $|\det N| \ge \bar{c}$ holds for the matrix N whose columns consist of the unit vectors $\boldsymbol{l}_i$, $i \in I_d$, along the outgoing edges from this vertex. The lengths of the edges corresponding to the vectors $\boldsymbol{l}_i$ are the element size parameters and are denoted by h_i. The degrees of freedom for the BDM interpolation operator I_k^{BDM} of order k on simplices can be found in [8, p. 59].

The error estimates for the BDM interpolation depending on the geometric regularity are given by the two following theorems, cf. [2, Theorems 4.3, 4.4] and [1, Theorems 6.2, 6.3].

Theorem 1. *Let a simplicial element T satisfy the regular vertex property with constant $\bar{c}$. Then for $k \ge 1$, $0 \le m \le k$ and $\boldsymbol{v} \in W^{m+1,p}(T)$, $1 \le p \le \infty$, the estimate*

$$\left\| \boldsymbol{v} - I_k^{\text{BDM}} \boldsymbol{v} \right\|_{0,p,T} \lesssim \sum_{|\boldsymbol{\alpha}| = m+1} h^{\boldsymbol{\alpha}} \left\| D_{\boldsymbol{l}}^{\boldsymbol{\alpha}} \boldsymbol{v} \right\|_{0,p,T} + h_T \sum_{|\boldsymbol{\alpha}| = m} h^{\boldsymbol{\alpha}} \left\| D_{\boldsymbol{l}}^{\boldsymbol{\alpha}} \text{div} \, \boldsymbol{v} \right\|_{0,p,T}$$

holds, the constant only depends on $\bar{c}$ and k. Here

$$h_T = \text{diam} \, T, \, h^{\boldsymbol{\alpha}} = \prod_{j \in I_d} h_j^{\alpha_j} \quad and \quad D_{\boldsymbol{l}}^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial_{\boldsymbol{l}_1}^{\alpha_1} \cdots \partial_{\boldsymbol{l}_d}^{\alpha_d}}.$$

Theorem 2. *Let a simplicial element T satisfy the maximum angle condition with constant $\bar{\phi}$. Then for $k \ge 1$, $0 \le m \le k$ and $\boldsymbol{v} \in W^{m+1,p}(T)$, $1 \le p \le \infty$, the estimate*

$$\left\| \boldsymbol{v} - I_k^{\text{BDM}} \boldsymbol{v} \right\|_{0,p,T} \lesssim h_T^{m+1} \left\| D^{m+1} \boldsymbol{v} \right\|_{0,p,T}$$

holds, the constant only depends on $\bar{\phi}$ and k. The notation D^n means the sum of the absolute values of all derivatives of order n.
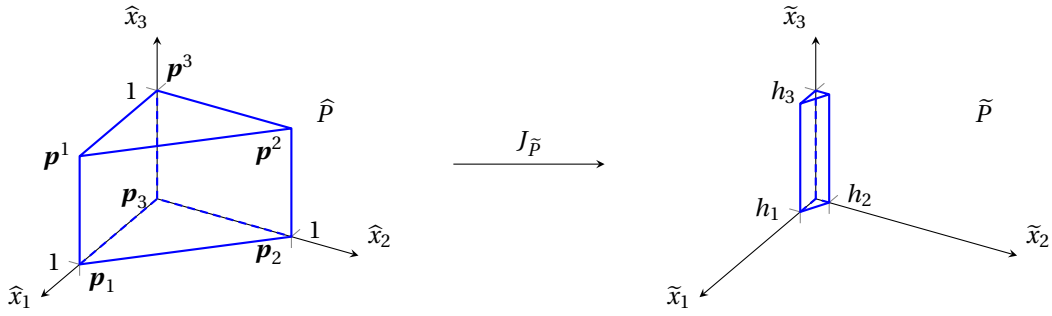
Figure 1. Reference prism $\widehat{P}$ with vertex numbering, and a transformed prism of the reference family $\mathscr{R}_P$.

3. Error estimates on anisotropic prisms

Similar results can be achieved for triangular prisms with some modifications. The prismatic reference element $\widehat{P}$ with the notation for the vertices is given on the left hand side of Figure 1.

The element $\widehat{P}$ is transformed to an element $\widetilde{P}$ of the reference family $\mathscr{R}_P$ by the transformation $\widetilde{\boldsymbol{x}} = J_{\widetilde{P}}\widehat{\boldsymbol{x}}$, where

$$J_{\widetilde{P}} = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}, \tag{1}$$

see Figure 1. The vertical facet of $\widehat{P}$ opposite of the vertices $\boldsymbol{p}_i$ and $\boldsymbol{p}^i$ is denoted by e_i, the horizontal facet at $\widehat{x}_3 = 0$ by e_b and the one at $\widehat{x}_3 = 1$ by e_t. The facet normals of $\widehat{P}$ are thus given by

$$\boldsymbol{n}_{\widehat{P},e_1} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{n}_{\widehat{P},e_2} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{n}_{\widehat{P},e_3} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{n}_{\widehat{P},e_b} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{n}_{\widehat{P},e_t} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The local BDM interpolation operator I_k^{BDM} of order k on a prism P maps into the space $\boldsymbol{P}_{k,k}(P)$ of vector valued polynomials of total degree k in x_1 and x_2 and degree k in x_3. It is defined by the functionals, see [8, p. 64],

$$\int_{e_i} I_k^{\mathrm{BDM}}\boldsymbol{v} \cdot \boldsymbol{n}_{P,e_i} z \,\mathrm{d}\boldsymbol{s} = \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n}_{P,e_i} z \,\mathrm{d}\boldsymbol{s} \qquad \forall\, z \in P_k(e_i), \quad i \in \{b,t\}, \tag{2a}$$

$$\int_{e_i} I_k^{\mathrm{BDM}}\boldsymbol{v} \cdot \boldsymbol{n}_{P,e_i} z \,\mathrm{d}\boldsymbol{s} = \int_{e_i} \boldsymbol{v} \cdot \boldsymbol{n}_{P,e_i} z \,\mathrm{d}\boldsymbol{s} \qquad \forall\, z \in Q_k(e_i), \quad i \in I_3, \tag{2b}$$

$$\int_P \left(I_k^{\mathrm{BDM}}\boldsymbol{v}\right)_3 z_3 \,\mathrm{d}\boldsymbol{x} = \int_P v_3 z_3 \,\mathrm{d}\boldsymbol{x} \qquad \forall\, z_3 \in P_{k,k-2}(P), \tag{2c}$$

$$\int_P \left(I_k^{\mathrm{BDM}}\boldsymbol{v}\right)_1 z_1 + \left(I_k^{\mathrm{BDM}}\boldsymbol{v}\right)_2 z_2 \,\mathrm{d}\boldsymbol{x} = \int_P v_1 z_1 + v_2 z_2 \,\mathrm{d}\boldsymbol{x} \qquad \forall\, (z_1, z_2) \in \mathscr{P}_{k-1,k}(P), \tag{2d}$$

where $P_k(e)$ is the space of polynomials with maximal degree k, $Q_k(e)$ is the space of polynomials with degree k in each of the dimensions of the facet e, $P_{m,n}(P)$ is the space of polynomials with total degree m in x_1 and x_2, and degree n in x_3. The space $\mathscr{P}_{m,n}(P)$ consists of pairs of polynomials with degree n in x_3, which are for fixed x_3, i.e., on the triangle T_{x_3}, in the space $\boldsymbol{N}_m(T_{x_3}) = \boldsymbol{P}_{m-1}(T_{x_3}) \oplus \boldsymbol{S}_m(T_{x_3})$, where $\boldsymbol{S}_m(T_{x_3}) = \{\boldsymbol{p} \in \boldsymbol{P}_m(T_{x_3}) : \boldsymbol{p}(\boldsymbol{x}) \cdot \boldsymbol{x} = 0 \ \forall\, \boldsymbol{x} \in T_{x_3}\}$.

We first require the analog of [2, Lemma 3.1] for the prism reference element. The proof is omitted here for brevity since it follows along the same steps as the proof for the analogous lemma for simplices.

Lemma 3. *Let $\widehat{P}$ be the reference element from Figure 1, $\widehat{f}_j \in L^p(e_j)$, $j \in I_2$, $\widehat{f}_3 \in L^p(e_b)$, $1 \le p \le \infty$, and*

$$\widehat{\boldsymbol{u}}(\widehat{\boldsymbol{x}}) = \left(\widehat{f}_1(\widehat{x}_2, \widehat{x}_3), 0, 0\right)^T, \qquad \widehat{\boldsymbol{v}}(\widehat{\boldsymbol{x}}) = \left(0, \widehat{f}_2(\widehat{x}_1, \widehat{x}_3), 0\right)^T, \qquad \widehat{\boldsymbol{w}}(\widehat{\boldsymbol{x}}) = \left(0, 0, \widehat{f}_3(\widehat{x}_1, \widehat{x}_2)\right)^T.$$

Then there are functions $\widehat{q}_j \in P_k(e_j)$, $j \in I_2$, $\widehat{q}_3 \in P_k(e_b)$, so that

$$\widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{u}}(\widehat{\boldsymbol{x}}) = \left(\widehat{q}_1(\widehat{x}_2, \widehat{x}_3), 0, 0\right)^T, \quad \widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{v}}(\widehat{\boldsymbol{x}}) = \left(0, \widehat{q}_2(\widehat{x}_1, \widehat{x}_3), 0\right)^T, \quad \widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{w}}(\widehat{\boldsymbol{x}}) = \left(0, 0, \widehat{q}_3(\widehat{x}_1, \widehat{x}_2)\right)^T.$$

With this lemma, we get the stability estimate on the reference element.

Lemma 4. *Let $\widehat{\boldsymbol{u}} \in W^{1,p}(\widehat{P})$, $1 \le p \le \infty$. Then we have the estimates*

$$\left\|\left(\widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{u}}\right)_i\right\|_{0,p,\widehat{P}} \lesssim \|\widehat{u}_i\|_{1,p,\widehat{P}} + \left\|\widehat{\mathrm{div}}\,\widehat{\boldsymbol{u}}\right\|_{0,p,\widehat{P}} + \left\|\frac{\partial \widehat{u}_3}{\partial \widehat{x}_3}\right\|_{0,p,\widehat{P}}, \qquad i \in I_2,$$

$$\left\|\left(\widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{u}}\right)_3\right\|_{0,p,\widehat{P}} \lesssim \|\widehat{u}_3\|_{1,p,\widehat{P}} + \left\|\widehat{\mathrm{div}}\,\widehat{\boldsymbol{u}}\right\|_{0,p,\widehat{P}}.$$

Proof. The proofs for $i \in I_2$ are analogous so we only show the details for the first and third components, starting with $i = 1$. Let

$$\widehat{\boldsymbol{u}}_* = \begin{pmatrix} 0 \\ \widehat{u}_2(\widehat{x}_1, 0, \widehat{x}_3) \\ \widehat{u}_3(\widehat{x}_1, \widehat{x}_2, 0) \end{pmatrix}, \qquad\qquad \widehat{\boldsymbol{v}} = \widehat{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_* = \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 - \widehat{u}_2(\widehat{x}_1, 0, \widehat{x}_3) \\ \widehat{u}_3 - \widehat{u}_3(\widehat{x}_1, \widehat{x}_2, 0) \end{pmatrix}.$$

The previous Lemma 3 thus yields $(\widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{v}})_1 = (\widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{u}})_1$, and it holds $\widehat{\mathrm{div}}\,\widehat{\boldsymbol{v}} = \widehat{\mathrm{div}}\,\widehat{\boldsymbol{u}} - \widehat{\mathrm{div}}\,\widehat{\boldsymbol{u}}_* = \widehat{\mathrm{div}}\,\widehat{\boldsymbol{u}}$. We define the two functions

$$\widehat{\boldsymbol{v}}_* = \begin{pmatrix} 0 \\ \widehat{x}_2 \widehat{q}_2 \\ \widehat{x}_3 \widehat{q}_3 \end{pmatrix}, \qquad\qquad \widehat{\boldsymbol{w}} = \widehat{\boldsymbol{v}} - \widehat{\boldsymbol{v}}_* = \begin{pmatrix} \widehat{v}_1 \\ \widehat{v}_2 - \widehat{x}_2 \widehat{q}_2 \\ \widehat{v}_3 - \widehat{x}_3 \widehat{q}_3 \end{pmatrix},$$

where $\widehat{q}_2 \in P_{k-1,k}(\widehat{P})$, $\widehat{q}_3 \in P_{k,k-1}(\widehat{P})$, so that

$$\int_{\widehat{P}} \widehat{w}_2 z \,\mathrm{d}\boldsymbol{x} = \int_{\widehat{P}} (\widehat{v}_2 - \widehat{x}_2 \widehat{q}_2) z \,\mathrm{d}\boldsymbol{x} = 0 \qquad \forall\, z \in P_{k-1,k}(\widehat{P}),$$

$$\int_{\widehat{P}} \widehat{w}_3 z \,\mathrm{d}\boldsymbol{x} = \int_{\widehat{P}} (\widehat{v}_3 - \widehat{x}_3 \widehat{q}_3) z \,\mathrm{d}\boldsymbol{x} = 0 \qquad \forall\, z \in P_{k,k-1}(\widehat{P}).$$

This means that the functions $\widehat{q}_2$ and $\widehat{q}_3$ are well defined. Since $\widehat{\boldsymbol{v}}_* \in \boldsymbol{P}_{k,k}(\widehat{P})$ it follows that

$$\widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{v}}_* = \widehat{\boldsymbol{v}}_* \quad \text{and thus} \quad \left(\widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{w}}\right)_1 = \left(\widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{v}}\right)_1 = \left(\widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{u}}\right)_1.$$

The interpolated function $\widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{w}} = \widehat{\boldsymbol{t}} = (\widehat{t}_1, \widehat{t}_2, \widehat{t}_3)^T$ is then defined by the relations, see (2),

$$\int_{e_b} \widehat{t}_3 z \,\mathrm{d}\boldsymbol{s} = \int_{e_b} \widehat{w}_3 z \,\mathrm{d}\boldsymbol{s} = 0 \qquad\qquad\qquad \forall\, z \in P_k(e_b),$$

$$\int_{e_t} \widehat{t}_3 z \,\mathrm{d}\boldsymbol{s} = \int_{e_t} \widehat{w}_3 z \,\mathrm{d}\boldsymbol{s} = \int_{e_t} \widehat{w}_3 z \,\mathrm{d}\boldsymbol{s} - \int_{e_b} \widehat{w}_3 z \,\mathrm{d}\boldsymbol{s} = \int_{\widehat{P}} \frac{\partial \widehat{w}_3}{\partial \widehat{x}_3} z \,\mathrm{d}\boldsymbol{x} \quad \forall\, z \in P_k(e_t),$$

$$\int_{e_1} \widehat{t}_1 z \,\mathrm{d}\boldsymbol{s} = \int_{e_1} \widehat{w}_1 z \,\mathrm{d}\boldsymbol{s} \qquad\qquad\qquad\qquad \forall\, z \in Q_k(e_1),$$

$$\int_{e_2} \widehat{t}_2 z \,\mathrm{d}\boldsymbol{s} = \int_{e_2} \widehat{w}_2 z \,\mathrm{d}\boldsymbol{s} = 0 \qquad\qquad\qquad \forall\, z \in Q_k(e_2),$$

$$\int_{e_3} (\widehat{t}_1 + \widehat{t}_2) z \,\mathrm{d}\boldsymbol{s} = \int_{e_3} (\widehat{w}_1 + \widehat{w}_2) z \,\mathrm{d}\boldsymbol{s} \qquad\qquad \forall\, z \in Q_k(e_3),$$

$$\int_{\widehat{P}} \widehat{t}_3 z_3 \,\mathrm{d}\boldsymbol{x} = \int_{\widehat{P}} \widehat{w}_3 z_3 \,\mathrm{d}\boldsymbol{x} = \int_{\widehat{P}} (\widehat{v}_3 - \widehat{x}_3 \widehat{q}_3) z_3 \,\mathrm{d}\boldsymbol{x} = 0 \qquad \forall\, z_3 \in P_{k,k-2}(\widehat{P}),$$

$$\int_{\widehat{P}} \widehat{t}_1 z_1 + \widehat{t}_2 z_2 \,\mathrm{d}\boldsymbol{x} = \int_{\widehat{P}} \widehat{w}_1 z_1 + \widehat{w}_2 z_2 \,\mathrm{d}\boldsymbol{x} = \int_{\widehat{P}} \widehat{w}_1 z_1 \,\mathrm{d}\boldsymbol{x} \qquad \forall\, (z_1, z_2) \in \mathscr{P}_{k-1,k}(\widehat{P}),$$

where the definitions of $\widehat{q}_2$, $\widehat{q}_3$, and that $\widehat{w}_2|_{e_2} \equiv 0$, $\widehat{w}_3|_{e_b} \equiv 0$ were used. Additional computations for the relation on e_3 yield

$$
\begin{aligned}
\frac{1}{\sqrt{2}} \int_{e_3} (\widehat{t}_1 + \widehat{t}_2) z \, \mathrm{d}s &= \frac{1}{\sqrt{2}} \int_{e_3} (\widehat{w}_1 + \widehat{w}_2) z \, \mathrm{d}s = \int_{e_3} \widehat{\boldsymbol{w}} \cdot \boldsymbol{n}_{e_3} \, \mathrm{d}s \\
&= \int_{\widehat{P}} \left(\frac{\partial \widehat{w}_1}{\partial \widehat{x}_1} + \frac{\partial \widehat{w}_2}{\partial \widehat{x}_2} \right) z \, \mathrm{d}\boldsymbol{x} + \int_{\widehat{P}} \widehat{w}_1 \frac{\partial z}{\partial \widehat{x}_1} \, \mathrm{d}\boldsymbol{x} - \int_{\partial \widehat{P} \setminus e_3} \widehat{\boldsymbol{w}} \cdot \boldsymbol{n}_{\partial \widehat{P}} z \, \mathrm{d}s \\
&= \int_{\widehat{P}} \left(\widehat{\mathrm{div}} \, \widehat{\boldsymbol{w}} - 2 \frac{\partial \widehat{w}_3}{\partial \widehat{x}_3} \right) \mathrm{d}\boldsymbol{x} + \int_{\widehat{P}} \widehat{w}_1 \frac{\partial z}{\partial \widehat{x}_1} \, \mathrm{d}\boldsymbol{x} - \int_{e_1} \widehat{w}_1 z \, \mathrm{d}s.
\end{aligned}
$$

Thus, the terms

$$
\int_{e_1} \widehat{w}_1 z \, \mathrm{d}s = \int_{e_1} \widehat{u}_1 z \, \mathrm{d}s, \qquad\qquad \int_{\widehat{P}} \widehat{w}_1 z \, \mathrm{d}\boldsymbol{x} = \int_{\widehat{P}} \widehat{u}_1 z \, \mathrm{d}\boldsymbol{x},
$$

$$
\int_{\widehat{P}} \frac{\partial \widehat{w}_3}{\partial \widehat{x}_3} z \, \mathrm{d}\boldsymbol{x} = \int_{\widehat{P}} \frac{\partial \widehat{u}_3 - \widehat{x}_3 \widehat{q}_3}{\partial \widehat{x}_3} z \, \mathrm{d}\boldsymbol{x}, \qquad\qquad \int_{\widehat{P}} (\widehat{\mathrm{div}} \, \widehat{\boldsymbol{w}}) z \, \mathrm{d}\boldsymbol{x} = \int_{\widehat{P}} \widehat{\mathrm{div}} \, (\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{v}}_*) z \, \mathrm{d}\boldsymbol{x}
$$

define the interpolant. We get the desired estimate

$$
\left\| \left(\widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{u}} \right)_1 \right\|_{0,p,\widehat{P}} \lesssim \| \widehat{u}_1 \|_{1,p,\widehat{P}} + \left\| \widehat{\mathrm{div}} \, \widehat{\boldsymbol{u}} \right\|_{0,p,\widehat{P}} + \left\| \frac{\partial \widehat{u}_3}{\partial \widehat{x}_3} \right\|_{0,p,\widehat{P}}
$$

using a trace theorem and steps similar to those in the proof of [1, Lemma 3.3] to estimate the terms involving $\widehat{x}_3 \widehat{q}_3$ and $\widehat{\mathrm{div}} \, \widehat{\boldsymbol{v}}_*$. For the third component, with the definitions

$$
\widehat{\boldsymbol{v}} = \widehat{\boldsymbol{u}} - \widehat{\boldsymbol{u}}_* = \begin{pmatrix} \widehat{u}_1 - \widehat{u}_1(0, \widehat{x}_2, \widehat{x}_3) \\ \widehat{u}_2 - \widehat{u}_2(\widehat{x}_1, 0, \widehat{x}_3) \\ \widehat{u}_3 \end{pmatrix}, \qquad \widehat{\boldsymbol{v}}_* = \begin{pmatrix} \widehat{x}_1 \widehat{q}_1 \\ \widehat{x}_2 \widehat{q}_2 \\ 0 \end{pmatrix}, \qquad \widehat{\boldsymbol{w}} = \widehat{\boldsymbol{v}} - \widehat{\boldsymbol{v}}_* = \begin{pmatrix} \widehat{v}_1 - \widehat{x}_1 \widehat{q}_1 \\ \widehat{v}_2 - \widehat{x}_2 \widehat{q}_2 \\ \widehat{v}_3 \end{pmatrix},
$$

where the functions $\widehat{q}_j \in P_{k-1,k}(\widehat{P})$, $j \in I_2$, are now defined by

$$
\int_{\widehat{P}} \widehat{w}_j z \, \mathrm{d}\boldsymbol{x} = \int_{\widehat{P}} \left(\widehat{v}_j - \widehat{x}_j \widehat{q}_j \right) z \, \mathrm{d}\boldsymbol{x} = 0 \qquad \forall \, z \in P_{k-1,k}(\widehat{P}),
$$

the relevant terms of the interpolation relations are

$$
\int_{e_b} \widehat{w}_3 z \, \mathrm{d}s = \int_{e_b} \widehat{u}_3 z \, \mathrm{d}s, \qquad\qquad \int_{e_t} \widehat{w}_3 z \, \mathrm{d}s = \int_{e_t} \widehat{u}_3 z \, \mathrm{d}s,
$$

$$
\int_{\widehat{P}} \widehat{w}_3 z_3 \, \mathrm{d}\boldsymbol{x} = \int_{\widehat{P}} \widehat{u}_3 z_3 \, \mathrm{d}\boldsymbol{x}, \qquad\qquad \int_{\widehat{P}} \widehat{\mathrm{div}} \, \widehat{\boldsymbol{w}} z \, \mathrm{d}\boldsymbol{x} = \int_{\widehat{P}} \widehat{\mathrm{div}} \, (\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{v}}_*) z \, \mathrm{d}\boldsymbol{x}.
$$

From here the same steps as for the first component yield the desired estimate. □

Using the transformation (1) we bring the stability estimate to an element of the reference family.

Lemma 5. *Let* $\widetilde{P} = J_{\widetilde{P}} \widehat{P} + \boldsymbol{x}_0$, $\boldsymbol{x}_0 \in \mathbb{R}^3$, *and* $\widetilde{\boldsymbol{v}} \in \boldsymbol{W}^{1,p}(\widetilde{P})$, $1 \le p \le \infty$. *Then on the prism* $\widetilde{P}$ *the estimate*

$$
\left\| \widetilde{I}_k^{\mathrm{BDM}} \widetilde{\boldsymbol{v}} \right\|_{0,p,\widetilde{P}} \lesssim \sum_{|\boldsymbol{\alpha}| \le 1} h^{\boldsymbol{\alpha}} \left\| D^{\boldsymbol{\alpha}} \widetilde{\boldsymbol{v}} \right\|_{0,p,\widetilde{P}} + h_{\widetilde{P}} \left\| \widetilde{\mathrm{div}} \, \widetilde{\boldsymbol{v}} \right\|_{0,p,\widetilde{P}} + (h_1 + h_2) \left\| \frac{\partial \widetilde{v}_3}{\partial \widetilde{x}_3} \right\|_{0,p,\widetilde{P}}
$$

holds, where $h_{\widetilde{P}} = \max\{h_1, h_2, h_3\}$.

Proof. Using Lemma 4 and the relations

$$
\begin{aligned}
\| \widetilde{\boldsymbol{w}} \|_{0,p,\widetilde{P}} &= \left(\int_{\widetilde{P}} \sum_{i \in I_d} |\widetilde{w}_i|^p \, \mathrm{d}\boldsymbol{x} \right)^{1/p} \le \left(\det J_{\widetilde{P}} \right)^{1/p} \sum_{i \in I_d} {}_i h^{-1} \left(\int_{\widehat{P}} |\widehat{w}_i|^p \, \mathrm{d}\boldsymbol{x} \right)^{1/p} \\
&= \left(\det J_{\widetilde{P}} \right)^{1/p} \sum_{i \in I_d} {}_i h^{-1} \| \widehat{w}_i \|_{0,p,\widehat{P}}, \\
&\quad \left(\det J_{\widetilde{P}} \right)^{1/p} \| \widehat{v}_i \|_{1,p,\widehat{P}} = {}_i h \sum_{|\boldsymbol{\alpha}| \le 1} h^{\boldsymbol{\alpha}} \left\| D^{\boldsymbol{\alpha}} \widetilde{v}_i \right\|_{0,p,\widetilde{P}},
\end{aligned}
$$

where $_i h = \prod_{j \in I_3 \setminus \{i\}} h_j$, we compute

$$\left\| \widetilde{I}_k^{\mathrm{BDM}} \widetilde{\boldsymbol{v}} \right\|_{0,p,\widetilde{P}} \le \left(\det J_{\widetilde{P}} \right)^{1/p} \sum_{i \in I_3} {}_i h^{-1} \left\| \left(\widehat{I}_k^{\mathrm{BDM}} \widehat{\boldsymbol{v}} \right)_i \right\|_{0,p,\widehat{P}}$$

$$\lesssim \left(\det J_{\widetilde{P}} \right)^{1/p} \left[\sum_{i \in I_3} {}_i h^{-1} \left(\| \widehat{v}_i \|_{1,p,\widehat{P}} + \left\| \widehat{\mathrm{div}} \, \widehat{\boldsymbol{v}} \right\|_{0,p,\widehat{P}} \right) + \frac{h_1 + h_2}{h_1 h_2 h_3} \left\| \frac{\partial \widehat{v}_3}{\partial \widehat{x}_3} \right\|_{0,p,\widehat{P}} \right]$$

$$= \sum_{i \in I_3} \left(\sum_{|\boldsymbol{\alpha}| \le 1} h^{\boldsymbol{\alpha}} \left\| D^{\boldsymbol{\alpha}} \widetilde{v}_i \right\|_{0,p,\widetilde{P}} + h_i \left\| \widetilde{\mathrm{div}} \, \widetilde{\boldsymbol{v}} \right\|_{0,p,\widetilde{P}} \right) + (h_1 + h_2) \left\| \frac{\partial \widetilde{v}_3}{\partial \widetilde{x}_3} \right\|_{0,p,\widetilde{P}}$$

$$\lesssim \sum_{|\boldsymbol{\alpha}| \le 1} h^{\boldsymbol{\alpha}} \left\| D^{\boldsymbol{\alpha}} \widetilde{\boldsymbol{v}} \right\|_{0,p,\widetilde{P}} + h_{\widetilde{P}} \left\| \widetilde{\mathrm{div}} \, \widetilde{\boldsymbol{v}} \right\|_{0,p,\widetilde{P}} + (h_1 + h_2) \left\| \frac{\partial \widetilde{v}_3}{\partial \widetilde{x}_3} \right\|_{0,p,\widetilde{P}}. \qquad \square$$

The estimate from the previous lemma can be brought to the general prism P where the transformation is assumed to be reasonable in a certain sense. The stability estimate on P can be used to get the interpolation error estimate by a Bramble–Hilbert type argument. The proofs for the stability and interpolation error estimates on the element P follow the same steps as their analogs on simplices, see [2, Theorems 3.5, 4.3] and cf. [1, Theorems 3.1, 6.2], which is why they are omitted for brevity.

Theorem 6. *Let P be a prism element that emerges by the affine transformation $\boldsymbol{x} = J_P \widetilde{\boldsymbol{x}}$, with $\| J_P \|_\infty, \left\| J_P^{-1} \right\|_\infty \le C$, of the element $\widetilde{P} \in \mathscr{R}_P$. Then for $\boldsymbol{v} \in W^{1,p}(P)$, $1 \le p \le \infty$, the estimate*

$$\left\| I_k^{\mathrm{BDM}} \boldsymbol{v} \right\|_{0,p,P} \lesssim \| \boldsymbol{v} \|_{0,p,P} + \sum_{j \in I_3} h_j \left\| \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{l}_j} \right\|_{0,p,P} + h_P \| \mathrm{div} \, \boldsymbol{v} \|_{0,p,P} + (h_1 + h_2) \left\| \frac{\partial v_3}{\partial \boldsymbol{l}_3} \right\|_{0,p,P}$$

is satisfied. The vectors $\boldsymbol{l}_j$ are the outgoing unit vectors along the edges adjacent to the transformed vertex $\boldsymbol{p}_i$.

Theorem 7. *Let a prism P satisfy the same condition as in Theorem 6. Then for $k \ge 1$, $0 \le m \le k$ and $\boldsymbol{v} \in W^{m+1,p}(P)$, $1 \le p \le \infty$, the estimate*

$$\left\| \boldsymbol{v} - I_k^{\mathrm{BDM}} \boldsymbol{v} \right\|_{0,p,P} \lesssim \sum_{|\boldsymbol{\alpha}| = m+1} h^{\boldsymbol{\alpha}} \left\| D_{\boldsymbol{l}}^{\boldsymbol{\alpha}} \boldsymbol{v} \right\|_{0,p,P} + h_P \sum_{|\boldsymbol{\alpha}| = m} h^{\boldsymbol{\alpha}} \left\| D_{\boldsymbol{l}}^{\boldsymbol{\alpha}} \mathrm{div} \, \boldsymbol{v} \right\|_{0,p,P}$$

$$+ (h_1 + h_2) \sum_{|\boldsymbol{\alpha}| = m} h^{\boldsymbol{\alpha}} \left\| \frac{\partial^{m+1} v_3}{\partial \boldsymbol{l}_1^{\alpha_1} \partial \boldsymbol{l}_2^{\alpha_2} \partial \boldsymbol{l}_3^{\alpha_3+1}} \right\|_{0,p,P}$$

holds and the constant only depends on k.

Remark 8. A limitation of our results is that the transformation from the general element to the reference element is affine, which excludes settings where, e.g., the rectangular faces are not flat or the triangular faces are not parallel in the general element. Such settings require further investigation.

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