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BRIEF REPORT

Synthetic Turbulence with Prescribed Probability Density Function and Application to Scalar Quantities Occurring in Reactive Flows

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Abstract
Based on a (synthetic) turbulent signal which obeys a Gaussian probability density function (PDF) together with some form of prescribed two-point statistics (i.e. integral length or time scales or turbulent energy spectrum), a simple algorithm is proposed to transform the original signal, such that it follows a new target PDF. It is shown that for many practical applications the transformation does not change the integral length or time scale more than a few per cent. The algorithm can be combined with any turbulence generator. It has applications for prescribing boundary or initial conditions of non-Gaussian signals in scale resolving simulations of turbulent flows, such as passive scalars like temperature, bounded passive scalars occurring in reactive flows or velocity signals close to walls.

Keywords Synthetic Inflow · Synthetic Initial Data · Prescribed Probability Density Function · Non-Gaussian turbulence

1 Introduction

The generation of synthetic turbulence has been a very active field of research during the last two decades (Wu 2017; Tabor and Baba-Ahmadi 2010). It can be used for prescribing turbulent boundary or initial conditions in scale resolving simulations (Wu 2017; Tabor and Baba-Ahmadi 2010; Klein et al. 2003), for benchmarking data analysis algorithms (e.g. Benedict et al. 2000) and for fundamental turbulence research (Klein 2024; Trenberth 1984; Klein and Germano 2018). Baik et al. (2024) have recently shown that turbulence generation can also be used to describe steady flow from packed beds or sinter materials. State of the art algorithms typically achieve to generate a signal with prescribed first and second order one-point statistics together with some form of two-point statistics, e.g. integral length or time scales or turbulent energy spectra. Because these algorithms are usually based on random numbers and because of the central limit theorem, it is expected that the probability distribution function of these pseudo turbulent signals follows a Gaussian

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probability density distribution. In particular this holds true for the digital filter based inflow generation (Klein et al. 2003) or for the ARMA (autoregressive moving average) process prescribed in Trenberth (1984).

According to She et al. (1988) it has been well verified that the probability distribution of the full velocity field of turbulent flows is often Gaussian and deviations from a Gaussian distribution become significant only at small scales. However, there can be considerable deviations from a Gaussian PDF e.g. close to walls or for bounded scalars (Pope 2000). In reactive turbulent flows the bimodal PDF plays an important role and also the β-PDF is often conveniently used to model the distribution of scalar quantities (Poinsot and Veynante 2005). An inverse Fourier Transform based method to generate a bimodal PDF has been suggested by Eswaran and Pope (1988), application to generation of initial data for reacting flow DNS can be found for example in Patel and Chakraborty (2016); Premkumar et al. 2024). A straightforward approach to obtain a log-normal distribution has been recently used by Premkumar et al. (2024) to study the effect on reactivity controlled compression ignition (RCCI). However, to the best knowledge of the author there is no method published in the open literature to generate (correlated) synthetic turbulence with an arbitrary but prescribed PDF.

2 Methodology

The digital filter based inflow generator (Klein et al. 2003) serves in this work as the baseline method to generate a correlated signal with prescribed integral scale. The idea of this method is to filter white noise random data, such that a target turbulent length or time scale is achieved. Because of the central limit theorem, it is expected, and can be easily verified, that the obtained signal obeys a Gaussian PDF. However, the methodology presented next is not limited to this specific turbulence generator.

A straightforward methodology in physical space to transform a standard normal Gaussian variable X into a variable with approximate bimodal PDF is the following transformation:

$$Y = 0.5 + 0.5 \tanh(\Theta \cdot X). \tag{1}$$

Here, the parameter Θ (taken as $\Theta = 2.5$ later on) determines how close the PDF is to a double delta distribution of the form $\alpha\delta(X) + \beta\delta(1 - x)$, where δ is the Dirac function. It is noted that the $\tanh(\cdot)$ functions in Eq. (1) serves the only purpose of mapping all real numbers X close to the values $\{-1{,}1\}$ such that, for high values of Θ, finally Y is close to $\{0{,}1\}$ and a bimodal distribution is approached. There is a strong similarity between the $\tanh(\cdot)$ and $\mathrm{erf}(\cdot)$ functions and in fact the first could be replaced by the latter in Eq. (1) which would make Eq. (1) similar to Eq. (2) below, but with a different scaling of the argument.

A very general methodology of transforming a standard normal (Gaussian) distributed random variable X into a random variable Y with a target cumulative distribution function F_Y is described next. Let Φ be the cumulative distribution function of the standard normal distribution which transforms X into a uniform distribution U.

$$U = \Phi(\mathrm{X}),\ \Phi(\mathrm{X}) = \frac{1}{2}\left[1 + \mathrm{erf}\left(X/\sqrt{2}\right)\right]. \tag{2}$$

This in turn can be transformed into Y using the inverse transformation sampling (Gentle 2003) (ITS) in the following manner:

$$Y = F_Y^{-1}(\Phi(X)) = F_Y^{-1}(U),\ \Phi(\mathrm{X}) = \frac{1}{2}\left[1 + \mathrm{erf}\left(X/\sqrt{2}\right)\right]. \tag{3}$$

The error function $\mathrm{erf}(\cdot)$ is typically a built-in default function in software for scientific computing. The inverse cumulative distribution function (ICDF) has been obtained in this work by spline interpolating table entries obtained with the "icdf" function of the software MATLAB. (Alternatively, it can be obtained by numerical integration of the PDF to yield the discrete version (X,Y) of the CDF and subsequently exchanging the roles of X and Y) The proposed methodology is applied to the two-parameter β distribution P_β which is often used to parameterise the behaviour of bounded scalars in the context of reactive flows (Poinsot and Veynante 2005):

$$P_\beta(u, \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1}. \tag{4}$$

The question addressed next is to what extent the transformation given in Eq. (3) changes the integral scales that were imposed in the first step. Before discussing the results, it is noted that the ICDF of the $\beta-$ PDF may feature high peaks towards unity for $\alpha \ll \beta$ (towards zero for $\beta \ll \alpha$). In this case a large number of interpolation points is needed to ensure good approximation of the ICDF or alternatively a non-uniform clustering of the interpolation points towards the boundary.

3 Results

Figure 1 shows 2D contours of a synthetic field u for the original Gaussian and four transformed fields with bimodal PDF (see Eq. 2), uniform PDF (see Eq. 3), β-PDF with $a = \beta = 2$ or with $a = 1, \beta = 10$, using ITS (see Eq. 4). The imposed integral scale corresponds to $1/8$ of the field of view. It becomes obvious from Fig. 1 and is intuitively clear, that the transformation changes the colour of the contour plot, but it will in principle not affect the structures. One-dimensional line plots of signals obtained with the same methodology are shown in Fig. 2a, while Fig. 2b. shows the corresponding PDFs of the signals. As a validation of the implementation of ITS Fig. 2c shows exemplarily a β-PDF for the parameters $a = 20, \beta = 2$ obtained from the algorithm compared with the analytical definition (Eq. 3). The agreement is nearly perfect.

Figure 3a shows the corresponding autocorrelation functions (see Pope 2000 for a definition) for the obtained signals compared to the baseline autocorrelation from the digital filter method (which is known to be of Gaussian shape (Klein et al. 2003)). The largest difference in integral scale is obtained for the bimodal distribution (11%) while in all the other cases the deviation is smaller than 4%. A more systematic numerical study for the β-PDF is shown in Fig. 3b, which depicts the ratio of the integral scale L_{11}^N from a variable obeying a normal distribution to that of a β-distributed variable L_{11}^β, for a range of parameters. As expected from Eq. 3 results are symmetric in α and β (apart from numerical and statistical inaccuracies). Most importantly $L_{11}^N/L_{11}^\beta \approx 1$ except for cases where either α or β is relatively small. Because the mean value μ of the β-PDF is given by $\mu = \alpha/(\alpha + \beta)$ and because of Eq. 4, this would correspond to monomodal distributions resembling a Dirac peak at either zero or one. The same analysis is repeated for a different inflow generator and very similar results are shown in the Appendix.

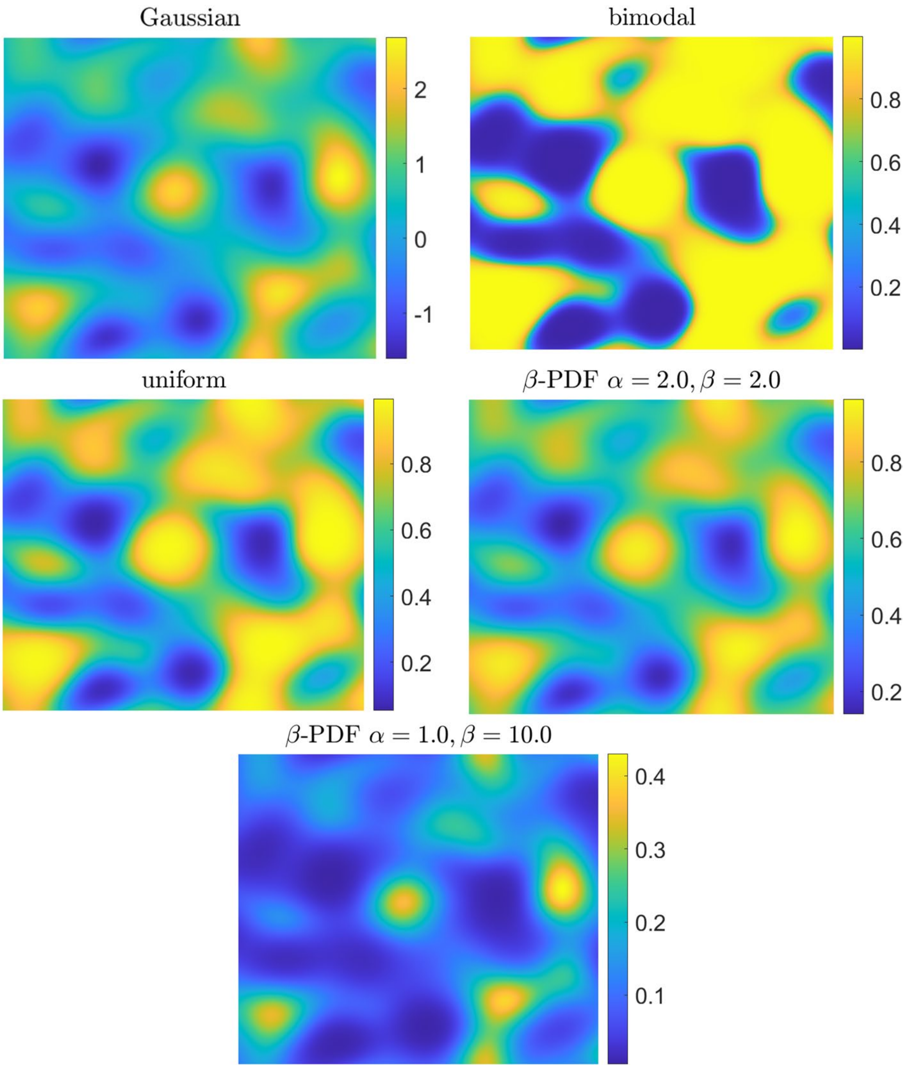

Fig. 1 2D contours of scalar field u for the original Gaussian and four transformed fields with bimodal PDF, uniform PDF, β-PDF with $a = \beta = 2$, β-PDF with $a = 1, \beta = 10$

4 Conclusions

A simple methodology is suggested, as an additional step of an existing (arbitrary) turbulence generator, to generate correlated, synthetic, pseudo-turbulent signals that obey a prescribed non-Gaussian PDF. It is shown that in most cases the transformation does not change the imposed integral scale to a large extent. The methodology is straightforward to implement and it is believed that it is in particular useful for prescribing initial or boundary conditions for scale resolving simulations of turbulent reactive flows.

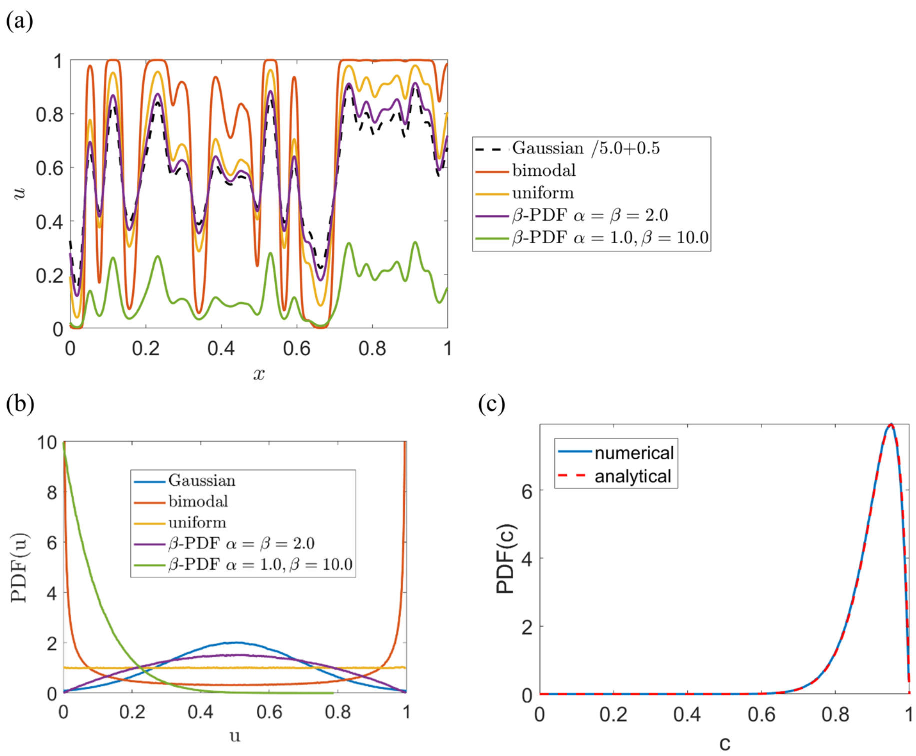

Fig. 2 **a** Section of 1D profiles of synthetic signals. The original Gaussian curve with $\mu = 0$ and $\sigma = 1$ is scaled to show it on the same diagram as the transformed signals with bimodal PDF, uniform PDF, β-PDF with $a = \beta = 2$, β-PDF with $a = 1, \beta = 10$; **b** PDFs of the five signals (note that the Gaussian is again scaled); **c** Comparison of PDF of transformed signal with the analytical shape of the β-PDF here shown for the parameters $a = 20, \beta = 2$

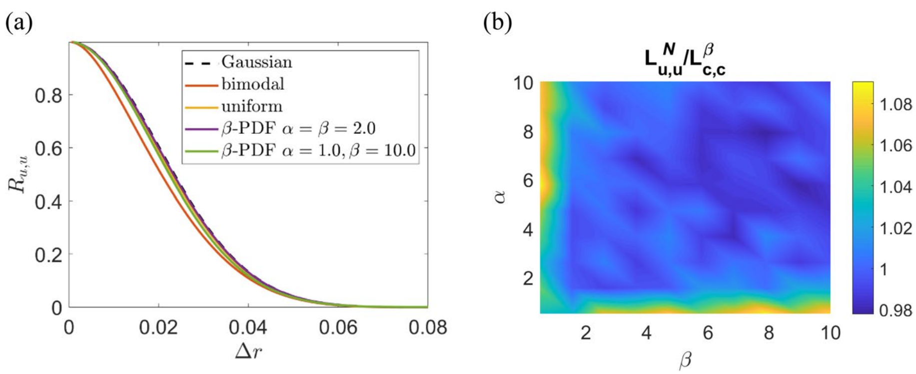

Fig. 3 **a** Longitudinal autocorrelation of the signals shown in Fig. 2; **b** Ratio of longitudinal integral scale of the original signal with normal (i.e. Gaussian) distribution and the transformed signal obeying a β-PDF for different parameters α, β

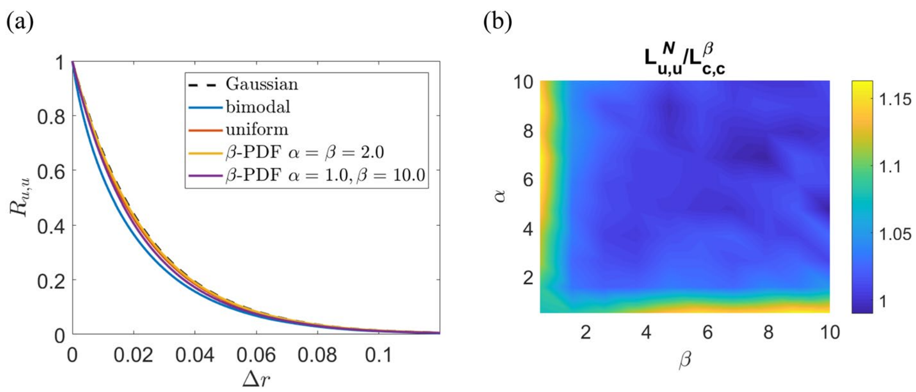

Fig. 4 **a** Longitudinal autocorrelation using an ARMA process with Gaussian PDF, bimodal PDF, uniform PDF, β-PDF with $a = \beta = 2$, β-PDF with $a = 1, \beta = 10$; **b** Ratio of longitudinal integral scale of the original signal with normal (i.e. Gaussian) distribution and the transformed signal obeying a β-PDF for different parameters α, β

Appendix

The earlier analysis has been repeated using the ARMA filter (Trenberth 1984), instead of the digital inflow generator, to create a random series. The results are depicted in Fig. 4. Note that the autocorrelation function of the ARMA process is exponentially decaying. The deviations are slightly larger compared to the digital filter case but still moderate.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no competing interests.

Ethical Approval No specific ethical approval is required for this work.

Informed Consent N/A

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