# Analysis and Optimal Design of Specific Trade Reduction Mechanisms

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## ABSTRACT

According to the London Gazette, auctions in the late 17th century were held in local taverns and coffee shops, with auction houses emerging in the 18th century. Today, auctions are held on a much larger scale with multiple buyers and sellers participating simultaneous. The design of these "double" auction mechanisms revolve around four main economic properties: Individual Rationality (IR), which incentivizes market participation; Incentive Compatibility (IC), which results in traders truthfully revealing their private information; ideal Efficiency, which processes all trades that result in net positive gains from trade; and Budget Balance (BB), which incentivizes the existence of the market itself by guaranteeing non-negative revenue.

The famous Myerson and Satterthwaite impossibility result proves that no single mechanism can achieve all four of these desired economic properties simultaneously. In a groundbreaking paper, McAfee introduced the Trade Reduction (TR) Mechanism, which circumvents this impossibility result. While ideal efficiency cannot be guaranteed in every case, the mechanism is able to either achieve ideal efficiency or reduce at most the least efficient trade. As the market grows large enough, this loss in ideal efficiency becomes negligible, allowing the mechanism to achieve asymptotic efficiency.

This Operations Research related study makes four key contributions, each focuses on the trade-off between two measures of performance: efficiency and revenue. The first contribution explores an alternative to the McAfee trade reduction mechanism by deliberately sacrificing both budget balance and ideal efficiency, enabling the optimization of both. The second proposes a pre-filter that detects inefficient traders and adjusts the mechanism to improve its resilience to market saturation. The third research contribution introduces optimal hyperparameters to generalize the McAfee mechanism and allow for distinct prices for buyers and sellers. This extends the budget balance requirement from merely ensuring non-negative revenue to generating positive revenue without sacrificing efficiency. The fourth and final contribution combines the pre-filter and the use of hyperparameters design one final mechanism that both generates a revenue in every outcome and is resilient to market saturation

The introduction of adjustable hyperparameters give greater control over the trade-off between efficiency and revenue. This mechanism is optimal under specific conditions, operates without requiring prior market information, and remains resilient even in markets heavily saturated with buyers or sellers.

Das Design von Doppelauktion-Mechanismen dreht sich um vier wesentliche wirtschaftliche Eigenschaften: Individuelle Rationalität (IR), die die Marktteilnahme anreizt; Anreizkompatibilität (IC), die dazu führt, dass Händler ihre privaten Informationen wahrheitsgemäß offenlegen; ideale Effizienz, die alle Handelsgeschäfte verarbeitet, die zu einem Nettogewinn führen; und Haushaltsgleichgewicht (BB), das die Existenz des Marktes selbst durch die Gewährleistung eines nichtnegativen Einnahmen anreizt. Das berühmte Unmöglichkeitsergebnis von Myerson und Satterthwaite beweist, dass kein einziges Mechanismus alle vier dieser gewünschten wirtschaftlichen Eigenschaften gleichzeitig erreichen kann. In einem bahnbrechenden Papier führte McAfee den Trade Reduction (TR) Mechanismus ein, der dieses Unmöglichkeitsergebnis umgeht. Während ideale Effizienz nicht in jedem Fall garantiert werden kann, ist der Mechanismus in der Lage, entweder ideale Effizienz zu erreichen oder höchstens den am wenigsten effizienten Handel zu reduzieren. Wenn der Markt groß genug wird, wird dieser Verlust an idealer Effizienz vernachlässigbar, wodurch der Mechanismus asymptotische Effizienz erreicht.

Diese Forschung führt das Design von vier neuen Mechanismen basierend auf der Trade Reduction Technik ein. Der erste, genannt der generalisierte McAfee Mechanismus, erweitert den ursprünglichen McAfee Trade Reduction Mechanismus durch die Einführung eines Hyperparameters, der den Bedarf an Trade Reduction in Märkten reduziert, die von einer Marktseite gesättigt sind. Die zweiten und dritten Mechanismen, genannt Epsilon- und Upsilon- Reduktionsmechanismen, bieten alternative Ansätze, um in jedem Ergebnis Einnahmen zu garantieren. Der vierte und letzte Mechanismus ist der Käufer- oder Verkäufer- Reduktionsmechanismus (BORS), der eine einfache Variation ohne Verwendung eines Hyperparameters einführt, um die Fälle zu erweitern, in denen ideale Effizienz garantiert ist.

Darüber hinaus beweist diese Forschung den notwendigen Kompromiss zwischen Determinismus und der wirtschaftlichen Eigenschaft eines starken Haushaltsgleichgewichts bei der Gestaltung eines Trade Reduction Mechanismus. Schließlich wird ein neuartiges Rahmenwerk eingeführt, das das bestehende und neu eingeführte Spektrum der Trade Reduction Mechanismen in einen 'Katalog' generalisiert. Dieses Rahmenwerk ermöglicht das optimale Design von Doppelauktion-Mechanismen basierend auf den gewünschten wirtschaftlichen Eigenschaften, dem Vorwissen und den Marktinformationen.

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# 1

# INTRODUCTION TO THE DESIGN OF AUCTION MECHANISMS

Auctions have existed since ancient times with reports from Herodotus indicating auctions were held in Babylon as early as 500 b.c. The "atrium auctionarium" used by the Roman Empire to liquidate property and estate goods is assumed to resemble the familiar ascending auction as the word "actus" in Latin means increasing. In the year 193 A.D. the entire Roman empire was put up for auction after it was sacked by the Praetorian Guard [24]. The classic auction theory textbook from Krishna [49] defines an auction as the process of procurement via competitive bidding.

Today, billion dollar industries use auctions to determine appropriate trading prices that allow for the distribution of goods and/or services when either the quantity is limited or the value of the goods and/or services are unknown [22]. This occurs often in emerging technology markets that have not yet been scaled for production. Emerging technologies are also difficult to value without a proper baseline with which to compare. Kumar et al. provide a comprehensive survey on the use of auctions in emerging technologies within the field of Operations Research [51].

## 1.1 AUCTIONS FOR EMERGING TECHNOLOGY: A BILLION DOLLAR INDUSTRY

The most common and largest example of the use of auctions in modern times are the reallocation of the electromagnetic spectrum due to the finite quantity. As technology becomes outdated it's use of the electromagnetic spectrum becomes under utilized creating a new market for emerging technology to acquire bandwidth on the spectrum. This reallocation was recently demonstrated with the introduction of 5G where Germany's Federal Network Agency auctioned off 41 blocks of the electromagnetic spectrum in the 2 GHz and 3.6 GHz bands [21]. The German 5G spectrum auction for 2 GHz and 3.6GHz resulted in a total sum of over 6.5 Billion Euro with four companies acquiring a total of 420 MHz [33].

The initial concept of implementing auction designs for the reallocation of the electromagnetic spectrum began in 1993 when the United States government through the Federal Communications Commission (FCC) decided to use auctions to allocate radio spectrum rights. Auctions were selected because they act as a significant mechanism to understand how both allocations and prices emerge.

New auction formats were designed by a pair of Stanford economists, Professor Dr. Paul Milgrom and Professor Dr. Robert Wilson. Their design promoted competition in industry and was lauded by the FCC. The New York Times even went as far as to call it the "the greatest auction in history" [83]. Their work led to the awarding of the 2020 Nobel prize in economic sciences [73]. The goal was to both encourage efficiency and maximize use of the spectrum as well as secure value for the US Treasury. Following the initial auction more than $300 billion of spectrum sales around the world have used some version of the rules introduced by the laureates [67].

The world's first two-sided (simultaneous bidding by both buyers and sellers) spectrum auction was held by the U.S. FCC in 2017 to reallocate the radio frequency spectrum from television broadcasters to meet the exploding mobile broadband demand for 5G and other wireless services [47]. Due to the size and complexity additional **Operations Research** methodologies were incorporated, including a customized series of optimization models, decompositions, cuts, heuristics, large neighborhood searches, and a portfolio of propositional satisfiability solvers whose parameters were determined by machine learning. The auction repurposed 84 megahertz of spectrum and generated gross revenue of nearly $20 billion, providing more than $10 billion in capital for the broadcast television industry and over $7 billion for federal deficit reduction [47].

The origins of auction theory often fall to the seminal article by WIlliam Vickery [94], however the first wave of research did not come until much later. There is no specific event signaling the founding of auction theory as a dedicated field of study; however, some of the first comprehensive publications dedicated solely to auction theory come from Milgrom [65], [66]. Meanwhile the first introductory survey to the field of auction theory confirming the existence of the field of study came from Klemperer [48].

At their core auctions are mechanisms for simultaneously allocating goods or services to buyers and/or sellers while also determining the prices to be paid and/or received via an explicit set of rules [27]. The unique design and combination of these rules significantly impact market outcomes.

In addition to major government spectrum auctions, other modern applications of auction theory are seen in online peer-to-peer (P2P) trading with services such as Ebay, Uber, and Airbnb. Applications in P2P trading can be found in AlAshery et al. [3] where the authors propose a P2P trading framework enabled by blockchain using the TR auction mechanism. In a different article the authors analyzes a power distribution system simulator interfaced with a P2P energy trading platform [43]. The platform similarly employs a blockchain-based distributed double auction trade mechanism.

This use of auction mechanism based blockchain technology arouses research on the combination with edge computing for device security, data security and forensics in the Internet of Things (IoT) because of its anonymity, non-tampering and traceability. The rapid development of big data, Internet of Things (IoT), artificial intelligence (AI), and many other technologies generates a massive amount of data and service requests at end devices. This requires the innovation of alternate computation solutions such as cloud and edge computing which have began to use auction and mechanism design approaches due to outstanding achievement of optimal allocation strategies while guaranteeing mutual satisfaction among servers and devices [80].

As a core technology to provide low latency and high communication for IoT, Edge Computing (EC) pulls computing resources from remote cloud centers to devices. Lio et at. analyze the application of blockchain using the TR based auction mechanism in MEC-IoT systems [55].

In Ma et al. [57] the authors exploit one of the key characteristics of the TR auction mechanism which is it's ability to guarantee truthfulness. They propose a truthful combinatorial double auction mechanism, which integrates the padding concept and the efficient pricing strategy to guarantee desirable properties in constrained EC environments. One estimate expects the edge computing industry expected to be worth $43 billion by 2027 [91].

Due to the significant financial impact of spectrum auctions worldwide, ongoing research in this area remains active. In an article written by Devi et al. [32] the authors deploy a Trade Reduction (TR) based auction model to efficiently and transparently allocate licensed unused bands in the electromagnetic spectrum to opportunistically improve spectrum utilization and alleviate imbalances.

Abbass et al. approach the same problem and are able to use their framework to account for the difference in priority for incumbent access (IA), primary access licensees (PAL), and general authorized access (GAA) [1]. A novel algorithm integrates the TR based auction that uses deep recurrent double Q-learning.

This shift toward edge and cloud computing reflects the need for faster, more localized data processing, especially as data-hungry applications like IoT and AI continue to expand. The need for immediate data processing at the edge is driving a paradigm shift in the infrastructure of digital enterprises. Auctions based on the trade reduction method offer an exciting application in cloud resource allocation. A dynamic TR auction-based resource allocation model is proposed by Li et al. to efficiently improve utilization for a cloud computing market [53].

A study by Alahdadi et al. proposes a TR based auction mechanism to counter the tendency of cloud users to bid lower prices while cloud service providers bid higher price than their true values leading to catastrophic market consequences [4].

Motivated by the wisdom of Immanuel Kant "sapere aude", which means "dare to know" in his easy "Answering the Question: What is Enlightenment?" [46], this research focuses on **optimizing** the trade off space between market efficiency and revenue, which is a central concern in mechanism design within operations research. This optimization is done subject to various constraints, foundational to auction theory and mechanism design.

Auction mechanism are designed around holding four key economic properties that 1. ensure participates, known as traders, are incentived to join the market by guaranteeing non-negative payoffs; 2. incentive traders to truthfully reveal their private information; 3. clear the market of all trades that are appealing to both the buyer and seller side of the market; and 4. guarantees a non-negative revenue to incentive the existence of the market itself.

Unique alternatives to classic auction mechanisms are proposed in this research to balance economic properties with the often competing objectives of revenue and efficiency.

By introducing novel methods for hyper parameter selection and applying pre-filters, the study explores how mechanisms can be fine-tuned to achieve greater efficiency in smaller markets as well as those prone to saturation, while still generating consistent revenue. These contributions provide practical tools for mechanism designers operating in complex, real-world environments such as cloud and edge computing auctions, where efficiency and revenue considerations must be carefully managed to ensure both scalability and sustainability.

This is particularly relevant as the United States Air Force is undergoing a department wide data modernization effort [2]. While traditional on-premises computing provides full control over systems and infrastructure, cloud services such as IaaS, CaaS, PaaS, FaaS, and SaaS offer varying levels of abstraction and automation, reducing the operational burden and allowing the department to scale resources and optimize costs more effectively.

Of these five models, the U.S. Department of the Air Force (DAF) has adopted the SaaS model to meet it's data needs [2]. The allows the DAF to protect the intellectual property of it's members and maintain ownership of data, architecture and modular design of software. This provides a cost-effective and stress-free solution to a wide range of military application on both unclassified and classified networks.

Auction-based cloud resource management offers a versatile and highly effective approach to managing cloud resources. Through dynamic pricing, efficient resource allocation, and the ability to enforce market structures, cloud providers can balance supply and demand, avoid resource waste, and encourage rational and strategic behavior among participants. This approach ensures that cloud services remain cost-effective, scalable, and highly responsive to the needs of a growing and dynamic market.

## 1.2 GLOSSARY

This section provides a glossary of the notation used in this research.

| Set Notation | Definition | Section |
|---|---|---|
| $\mathbf{I}$ | Players | 1.3 |
| $\mathbf{Y}$ | All unique player strategy profiles | 1.3 |
| $\mathbf{O}$ | Finite possible outcome | 1.3 |
| $\mathbf{N}$ | Buyers | 2.1.1 |
| $\mathbf{M}$ | Sellers | 2.1.1 |
| $\mathbf{T} = \mathbf{T}_b \dot{\cup} \mathbf{T}_s$ | Winning buyer's and seller's bids | 2.1.2 |
| $\overline{\mathbf{T}}$ | losing bids | 2.1.2 |
| $\mathbf{\Gamma} = \mathbf{\Gamma}_b \dot{\cup} \mathbf{\Gamma}_s$ | Ideally efficient trader bids | 2.2.4 |
| $\overline{\mathbf{\Gamma}}$ | Inefficient trader bids | 2.2.4 |
| Vector Notation | | |
| $\nu = (\nu_1, \nu_i, ..., \nu_q) \forall i \in \mathbf{I}$ | Player strategies | 1.3 |
| $\mathbf{v} = (v_1, ..., v_n)$ | Buyer's valuation price | 2.1.1 |
| $\mathbf{b} = (b_1(v_1), ..., b_n(v_n)$ | Buyer's bids | 2.1.1 |
| $\mathbf{r} = (r_1, ..., r_m)$ | Seller's reservation price | 2.1.2 |
| $\mathbf{s} = (s_1(r_1), ..., s_m(r_m)$ | Seller's bids | 2.1.2 |
| Function Notation | | |
| $f : \mathbf{Y} \to \mathbf{O}$ | Map strategy to outcome | 1.3 |
| $u(O)$ | Utility function | 1.3 |
| $b_i(v_i) \forall i \in \mathbf{N}$ | Buyer's bid as a function of their private valuation | 2.1.1 |
| $s_j(r_j) \forall j \in \mathbf{M}$ | Seller's bid as a function of their private reservation price | 2.1.2 |
| $A : (\mathbf{b, s}) \to \mathbf{T} \dot{\cup} \overline{\mathbf{T}}$ | Allocation rule | 2.2 |
| $P : (\mathbf{b, s}) \to (\mathbf{p_b}, \mathbf{p_s})$ | Payment rule | 2.2 |
| $X(A, P)$ | Mechanism | 2.2 |
| Variable Notation | | |
| $g$ | Good/service | 1.3 |
| $p \in \mathbb{R}$ | Price | 1.3 |
| $\Pi$ | Auction's revenue | 2.1.3 |
| $\Omega$ | Auction's gains from trade (GFT) | 2.1.3 |
| $\overline{\Omega}$ | Auction's Social Welfare | 2.1.3 |
| $e$ | efficiency | 2.1.3 |

Table 1.1: Glossary

## 1.3 PRELIMINARY GAME THEORETIC CONCEPTS

This sections begins by defining game theory and preliminary concepts to include utility, strategies, and equilibria with a brief example. The section continues by showcasing various applications of auction mechanisms before concluding with the relation to auction theory as a game theoretic application of theoretical economics within operations research.

According to Maschler et al. auction theory serves as one of the most successful applications of game theory [59]. Game theory studies interactive situations among multiple individuals, called players, who make decisions that jointly determine the final outcome.

**Definition 1** : *Game Theory is the methodology of applying mathematical tools to model and analyze decision making involving several decision makers, known as players, where each decision affects the outcome for all.* [59]

Game theory frames these interactive situations involving the collective decisions of all players as a game to capture the decision space and boundaries of the game. A definition of a game-frame comes from Bonanno [18].

**Definition 2** : *A game-frame in strategic form is a list of four items (a quadruple)* $\langle \boldsymbol{I}, \mathbf{Y}, \boldsymbol{O}, f \rangle$ *where:*

- *A set of players* $\boldsymbol{I} = \{1, 2, ..., q\}$ *(*$q \geq 2$*)*
- *A set of all strategy profiles,* $\mathbf{Y} = (\mathrm{Y}_1, \mathrm{Y}_i, ..., \mathrm{Y}_q) \forall i \in \boldsymbol{I}$*, where an element of the set is a vector consisting of one strategy for each player,* $\nu = (\nu_1, \nu_i, ..., \nu_q) \forall i \in \boldsymbol{I}$*.*
- *A finite set of possible outcomes,* $\boldsymbol{O}$
- *A function,* $f : \mathbf{Y} \rightarrow \boldsymbol{O}$*, that maps an outcome with every strategy profile [18]*

In game theory the decision makers have their own set of unique preferences over the possible outcomes of the game. Suppose decision makers, also known as players, may collectively decide between a finite number, $K$, of possible outcomes denoted $\mathbf{O} = \{O_1, O_2, ..., O_K\}$, where $O_k$ denotes a specific outcome. Further, suppose each player $i$ has a preference relation, denoted $\succsim_i$ over the set of outcomes, so that $O_1 \succsim_i O_2$ indicates player $i$ either prefers outcome $O_1$ or is indifferent between them.

**Definition 3** : *An ordinal game in strategic form is a quintuple* $\langle \boldsymbol{I}, \mathbf{Y}, \boldsymbol{O}, f, (\succsim_1, ..., \succsim_q) \rangle$ *where:*

- *A game-frame is in strategic form,* $\langle \boldsymbol{I}, \mathbf{Y}, \boldsymbol{O}, f \rangle$*, according to Definition 2*
- *For every player i;* $\succsim_i$ *is a complete and transitive ranking of the set of outcomes* $\boldsymbol{O}$*. [18]*

Player preferences leads to a key concept in not only game theory but economic theory. This is the concept of utility.

**Definition 4** : ***Utility*** *is defined as the usefulness or enjoyment a consumer can get from a good or service* [17].

In other words, utility is as a measure of satisfaction. A quantitative representation of players' preferences over the various outcomes, called a utility function, was introduced in John von Neumann and Oscar Morgenstern's book, Theory of Games and Economic Behavior, originally published in 1944, then revised in 1947 [95]. This work is considered the foundation of game theory.

**Definition 5** : *A utility function, $u_i$, represents the preferences of player i as a real-valued function defined over the set of outcomes* $\mathbf{O}$ *according to equation 1.1.* [59]

$$u_i(O_1) \geq u_i(O_2) \iff O_1 \succsim_i O_2 \quad \forall O_1, O_2 \in \mathbf{O} \tag{1.1}$$

The utility function, as the name implies, is a function and can be thought of as an "index of satisfaction" where a higher value indicates a better or more preferred outcome [18].

Its use facilitates the application of mathematical tools in analyzing game situations with varying and uncertain outcomes and is a fundamental element of economics, to include not only game theory but also decision theory. The interactive nature of game theory is what distinguishes the field of study from standard decision theory which involves a single decision maker.

Utility can then be used to compare and relate different strategies in one of three ways by Definition 6.

Given the set of strategies $\mathbf{Y}$ in Definition 2 denote the set of all strategies excluding player $i$ as $\mathbf{Y}_{-i}$ with an element of on strategy for each player excluding player $i$ denoted $\nu_{-i}$.

**Definition 6** : *Given an ordinal game in strategic form, let i be a player with two possible strategies,* $a, b \in \mathrm{Y}_i$. *For Player i,*

- *a* ***strictly dominates*** *b if a gives player i a greater utility than that of b independent of the actions of all other players:* $u_i(a, \nu_{-i}) > u_i(b, \nu_{-i})$ $\forall \nu_{-i} \in \mathbf{Y}_{-i}$.
- *a* ***weakly dominates*** *b if a gives player i a greater than or equal to utility than that of b AND there exists at least one situation in which a gives player i a greater utility than that of b independent of the actions of all other players:* $u_i(a, \nu_{-i}) \geq u_i(b, \nu_{-i}) \quad \forall \nu_{-i} \in \mathbf{Y}_{-i}$ *and there exist a strategy* $\hat{g}_{-i} \in \mathbf{Y}_{-i}$ *such that* $u_i(a, \hat{g}_{-i}) > u_i(b, \hat{g}_{-i})$.
- *a* ***is equivalent to*** *b if a gives player i the same utility of that of b independent of the actions of all other players:* $u_i(a, \nu_{-i}) = u_i(b, \nu_{-i}) \quad \forall \nu_{-i} \in \mathbf{Y}_{-i}$. *[18]*

The relations in Definition 6 lead to the definition of dominant strategies which can either be strict or weak.

**Definition 7** : *Given an ordinal game in strategic form, let i be a player with one possible strategy, $a \in Y_i$. For Player i,*

- *a is a **strictly dominate strategy** if a strictly dominates every other strategy of player i.*
- *a is a **weakly dominate strategy** if, for every other strategy â of player i, either (1) a weakly dominates â or(2) a is equivalent to â [18]*

Comparing strategies allows for the definition of a foundational element of game theory: Equilibria.

**Definition 8** : *Given an ordinal game in strategic form, let $v = (v_1, v_2, ..., v_q)$ be a strategy profile.*

- *$v$ is a **strictly dominate strategy equilibrium** if every $v_i \quad \forall i \in \boldsymbol{I}$ is a strictly dominant strategy.*
- *$v$ is a **weakly dominate strategy equilibrium** if, both, every $v_i \quad \forall i \in \boldsymbol{I}$ is a weakly dominant strategy AND for at least one player $j$, $v_j$ is NOT a strictly dominant strategy. [18]*

The second point in Definition 8 is what distinguishes a weakly dominate strategy with a strictly dominant strategy. The outcome in which one player has a strictly dominant strategy while all others are weakly dominant is a strictly dominant strategy equilibrium.

Both strictly and weakly dominant strategy equilibrium fall into the more general definition of a Nash Equilibrium by Definition 9 as no one player has an incentive to deviate from their strategy.

**Definition 9** : *Given an ordinal game in strategic form with q players, a strategy profile $v^* \in \mathbf{Y}$ is a Nash equilibrium if all of the inequalities in equation 1.2 are satisfied.* [18]

$$u_i(v*) \geq u_i(v^*_{-i}, v_i) \quad \forall v_i \in \mathbf{Y}_i; \forall i \in \mathbf{I} \tag{1.2}$$

Consider the classic "Prisoner's Dilemma" in which two players are arrested by police and accused of committing a crime. The players are separated into different rooms and are given an opportunity to confess and cooperate with the investigation. Each player must decide whether to confess on the other player or to stay silent.

With two players making a choice between two strategies, four possible outcomes can occur. If both players stay silent then both will be charged with a minor crime and will only serve one year in prison. If both cooperate than they both be charged with a major crime, whoever they will get a reduced sentence and serve 5 years. If one player confesses while the other stays silent then the person cooperating will be set free leaving the silent player to serve a full 10 year sentence.

This game is presented in Table 1.2. Player 1 is considered the row player while player 2 acts as the column player. Each cell acts as one possible outcome showing the prison sentence as follows: (Player 1, Player 2).

| | | Player 2 | |
|---|---|---|---|
| | | Confess | Stay silent |
| Player 1 | Confess | (5,5) | (0,10) |
| | Stay silent | (10,0) | (1,1) |

Table 1.2: Prisoner's Dilemma

The Prisoner's Dilemma is formatted specifically to extract confessions. Regardless of the action of the row player it is best for the column player to confess, and vice versa. This results in both players confessing, despite the existence of an alternate outcome (in which both players stay silent) that is preferable to both players. The strategy to confess is said to dominate the strategy to stay silent. The prisoner's dilemma produces a strictly dominant-strategy equilibrium according to Definition 8 with the outcome in which both players confess. This is a Nash equilibrium by Definition 9 as neither player has an incentive to deviate, despite an alternate outcome existing which is better off for both players.

Extensive research has developed since the founding of game theory with a wide range of applications including, but not limited to the following list from Maschler et al. [59]:

- **Theoretical economics**. A market can be formulated as a game in which merchants sell goods or services to customers. Merchants can set prices for goods/services they offer and customer can shop for an acceptable price and quantity. Game theory models markets in an attempt to predict prices and study its relationship to the supply and demand. Auction theory exists as an application of game theory toward theoretical economics. Participants in an auction bid the prices they would be willing to buy and sell. Game theory is used to predict the bidding behavior as well as determine expected revenue the auction will produce according to the format used.
- **Political science**. Many states are governed through elected officials affiliated with political parties. An election can formulate as a game where the outcome is dependent upon any coalitions made within and between parties. If a coalition wins they may divide responsibilities of governing among many elected offices, for example parliamentary speaker and committee chairmanships. Game theory applies developed indices representing the power and influence of each party which can be used to predict and explain the division of responsibilities according to the results of an election [74]. Game theory can also provide insight into various voting mechanisms and the properties thereof.

- **Military applications**. Game theory can be used to model air to air combat maneuvers and missile flight trajectories. At an unclassified level, missile can take on various trajectories according to the environment in which they are deployed. Air combat engagements are often short lived and proper maneuvers are dependant on all other forces, both red and blue (threats and allies). Game theory can provide strategic thinking and insight to such situations to train pilots and provide vital information during mission planning. A fascinating contribution from Lt Col Dr. Mickelsen, United States Air Force, shows the effect of noise in various contests to include the classic Colonel Blotto game to account for the decentralization of forces [64]. Additional examples can be found in the other warfare domains in a survey by Ho et al. [44].
- **Inspection**. This refers to any situation in which a player can benefit from violating a law or agreement and the remaining players must monitor their behavior, thus acting as an "inspector". An example of where a game can be formulated this way is the relationship of actors in the nuclear enterprise. The International Atomic Energy Agency exists to enforce the Treaty on the Non-Proliferation of Nuclear Weapons through the inspection of nuclear facilities. Alternate individual examples include taxation audits or ticket inspection on public transportation. Meng et al. conducted a recent review of game theory applications toward safety management [63].
- **Biology**. Games can even be formulated where players are plants and animals. Flowers employ pollination strategies to attract insects through evolutionary development. The insects likewise must decide through evolution which flowers to visit. R. A. Fisher is considered the first to develop evolutionary game theory in 1930 (latest edition in 1999) by exploring the approximate equality of the sex ratio in mammals [37]. A game theoretic concept that uses Darwin's principle of "survival of the fittest" can be used to accurately explain various biological phenomena; this is the concept of Evolutionarily Stable Strategy (ESS) [5]. The main property of ESS states that a population where all players (e.g. animals/plants, etc.) play a the same stable strategy is resistant to the corruption of a smaller group who play an alternate strategy due to some mutation. For additional information see publications by Sandholm [84] and Broom and Rychtar [20].

This research considers applications in theoretical economics, specifically auction theory, which, according to Maschler, is a central element of modern economic theory [59].

Auction theory refers to players as traders who can take the form of either buyers or sellers. Auctions consist of buyer, sellers, or both depending on the auction format. Auctions exist to determine both the final allocation of some good/service as well as prices to be exchanged, denoted $p \in \mathbb{R}$.

**Definition 10** : *Denote a good or service as $g$. A trader is a player in an auction that can be defined as either a buyer interested in acquiring $g$ or a seller interested exchanging $g$ for a price $p$ who act as strategic utility maximizers* [67].

In an auction traders compete with one another via bids that act as messages indicating their willingness to pay. Traders submit bids to achieve their preferred outcome which consists of both the allocation of $g$ as well as the prices to be paid/received. Such a market exists as a result of buyers and sellers being in contact with one another enabling the exchange of goods and services to take place. From an economic perspective this can viewed in terms of supply and demand, where the intent is to facility the flow from supply to demand. Accomplishing this "clears" the market of potential trades. In auction theory market clearing is the process by which sellers representing supply is traded to buyers representing demand. An auction acts as that clearing process.

Taxonomy Relationships

| Game Theory Concepts | | Auction Theory Terms |
|---|---|---|
| Players | $\longleftrightarrow$ | Traders (buyers and/or sellers) |
| Strategy profiles | $\longleftrightarrow$ | Bids |
| Outcomes | $\longleftrightarrow$ | Allocation AND payments to be rendered for the goods/service |
| Function that maps strategies to outcomes | $\longleftrightarrow$ | Two interrelated functions (rules) that map bids to an allocation and payments |

Table 1.3: Taxonomy: game Vs. auction theory

Table 1.3 relates game theoretic concepts from Definition 2 to the taxonomy familiar in auction theory to be introduced in Chapter 2.

From an economics perspective auctions exists to clear markets by facilitating the flow from supply to demand. A supply curve can be generated via seller's bids, while a demand curve is created from buyer's bids. An auction implements rules with the goal of finding the point at which the supply meets demand to determine who engages in trade and at what price(s).

An example of a supply and demand curve is shown in Figure 1.1.

This example deals with a market in which five sellers, $m = 5$ (A,B,C,D,E) representing the supply, each have ten identical goods to sell with asking prices of \$10, \$15, \$20, \$25, and \$30. A trading price below \$10, would result in no sellers being interested in trade. A price between \$20 and \$25, would result in sellers A,B and C prepared to trade.

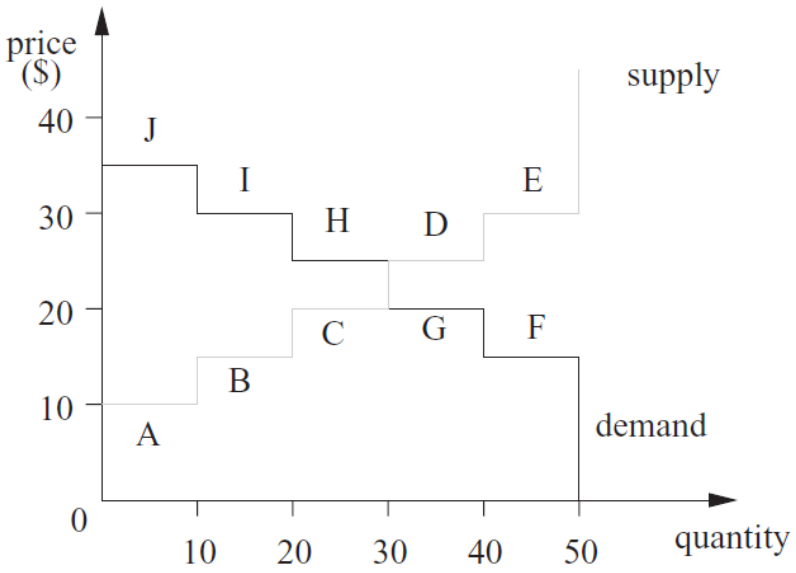



Figure 1.1: Supply and Demand curve

Similarly, five buyers, $n = 5$ (F, G, H, I, J) representing the demand are all interested in acquiring these goods at the various competitive pricing offers of $35, $30, $25, $20, and $15. A price above $35 while desirable for all five sellers would not leave a single buyer prepared to trade. According to the demand a price between $20 and $25, would result in three buyers H, I, and J prepared to make a trade.

Each of these constructed step-wise functions for supply and demand are known as the supply and demand curves. Where the supply meets the demand (any price between $20 and $25 say $23) is known as the equilibrium price. Once the price the set the market is then cleared by allowing the goods to flow from the supply to the demand and the cash to flow from the demand to the supply.

## 1.4 THESIS ORGANIZATION

This thesis is organized into six chapters. The first chapter provides the motivation and baselines various game theoretic concepts that relate to the theory of auctions and mechanism design. The chapter continues with the organization and concludes by presenting the research contributions.

The fundamentals and definition of notations are provided in chapter two. Section 2.1 makes the delineation between the familiar one-sided auction, in which an auctioneer sells of an item to a group of bidders and a two-sided auction in which a group of sellers aim to auction off items to a group of buyers that make up a market. A detailed description of mechanism design is then provided which introduces and defines the desired economic properties a mechanism can have along with the challenges in attaining them. The chapter concludes with the presentation of research questions.

Chapter 3 presents a deep dive into literature to discover each mechanism that attains the combination of properties concerned with in this research. The literature is organized chronologically to reveal the evolution of mechanisms that utilize the trade reduction technique.

Three research questions are addressed and four contributions are presented in Chapter 4. It begins with a section of design challenges and pitfalls and then presents four new mechanisms focused on the trade-off between efficiency and revenue. The final mechanism combines two previous contributions into a single mechanism. Newly introduced mechanisms serve as a generalization of existing mechanisms.

A discussion is provided in Chapter 5 that relates the novel findings to the emerging technology of cloud/edge computing and acknowledges the limitations in application while stating what is now possible. Chapter 6 provides a conclusion and outlines potential follow on efforts and future research.

## 1.5 RESEARCH CONTRIBUTION

This research focuses on trade reduction mechanisms, which originate from McAfee's seminal paper [60], due to their ability to circumvent the well-known impossibility result by Myerson and Satterthwaite [70]. The impossibility result demonstrates that no mechanism can simultaneously achieve four desired economic properties. These properties are individual rationality, incentive compatibility (truthfulness), budget balance, and ideal (perfect) efficiency. Trade reduction mechanisms address this limitation by recognizing that a minimal sacrifice in efficiency is necessary.

However, they optimize this loss so that efficiency approaches its ideal level as the market size becomes large, achieving what is known as asymptotic efficiency. In sufficiently large markets, the loss in ideal efficiency becomes negligible, thereby effectively circumventing the impossibility result.

Existing mechanisms, however, can struggle in buyer or seller saturated markets. A saturated market is one in which the number of buyers significantly exceeds the number of sellers, or vice versa.

This research makes four key contributions, each focuses on the trade-off between two measures of performance: efficiency and revenue.

The first contribution explores an alternative to the McAfee trade reduction mechanism by deliberately sacrificing both budget balance and ideal efficiency, enabling the **optimization** of both.

1. Section 4.1.3 presents the 'Alternate VCG' mechanism, which enhances efficiency benchmarks of existing mechanisms while maintaining a near-zero budget. It is resilient to market saturation and generates a slight positive or negative revenue.

This mechanism is named because it provides and alternative to the well studied Vickrey-Clarke-Groves (VCG) mechanism, named after their individual contributions [94], [28], and [42]. The VCG mechanism is individual rational, incentive compatibility, and ideally efficient, but runs a budget deficit.

The second contribution is concerned with the design of a mechanism that **maximizes efficiency** and is resilient to market saturation.

2. Section 4.2 enhances an existing mechanism by introducing a "pre-filter" that detects inefficient traders and adjusts the mechanism to improve its resilience to market saturation.

The "pre-filter" acts as an extension to a pair of existing mechanisms titled SBBA [87] and maintains the more strict *strong* budget balanced property. These new mechanisms answer the second research question and are the first mechanisms that are resilient to market saturation while maintaining the desired economic properties of individual rationality, incentive compatibility, budget balance, and are asymptotically efficient.

The third research contribution focuses on going beyond the budget balance requirement from merely ensuring non-negative revenue to guaranteeing positive revenue without sacrificing efficiency.

3. Section 4.3 introduces the design of a new trade-reduction mechanism that can guarantee a revenue in each and every outcome.

This contribution address the third research questions and is the first mechanism that is able to clear the market with ideal efficiency while guaranteeing a positive revenue.

The final contribution combines the pre-filter from the second contribution and the use of hyperparameters in the third contribution to design one final mechanism.

4. Section 4.4 introduces the design of a new trade-reduction mechanisms that can guarantee a revenue in each and every outcome and is resilient to market saturation

The novelty of this final mechanism, titled the "generalized trade reduction" mechanism, is that it generalizes the classic McAfee mechanism [60], a foundational model in auction theory. The introduction of adjustable hyperparameters give greater control over the trade-off space between efficiency and revenue, first characterized by Loertscher et al. [56]. Assuming the designer is unwilling to implement a mechanism that results in a budget deficit, in an increase in efficiency will require a transfer of revenue to both sides of the market.

Finally, the problem of proportionally due to market saturation has been investigated under the condition bids are made strategically [35]. This mechanism avoids strategic bidding by incentivizing traders to truthfully reveal their private valuations and remains resilient even in heavily saturated markets. The mechanism also operates without needing prior market information.

# 2

# FUNDAMENTALS AND NOTATION

This chapter defines the mathematical notation and components of auction design to include the delineation in auction types. The chapter first focuses on the various designs an auction can take and characteristics of the buyers and sellers that make up an auction. The chapter continuous to define the overall market, it's features, and the definition of a mechanism that used to process the flow of supply to demand. The chapter concludes by discussing various desired economic properties and the challenges involved in designing specific mechanism to achieve them.

## 2.1 AUCTION THEORY

Auctions can take many forms, ranging from simple one-sided auctions to more complex two-sided auctions like those considered in this work. Two-sided auctions are a generalization of one-sided auctions, involving both buyers and sellers simultaneously [61]. Auctions are typically tailored to the specific market in which they will operate. Some may be designed to favor buyers, while others may benefit sellers, depending on the market's objectives. Additionally, auctions can be structured to maximize revenue for the auctioneer or allow the surplus to remain with the participants, benefiting buyers and sellers directly. At their core, auctions serve two main purposes: determining the final allocation of goods or services and establishing the prices buyers will pay and sellers will receive [49].

### 2.1.1 *One Sided Auctions*

The most familiar auction format involves an auctioneer standing at the front of a room with a gavel iteratively raising the price of some good or service to be sold until no challengers remain, at which point the final remaining bidder wins the auction and pays the final price that was bid.

Consider a set $\mathbf{N}$ buyers interested in acquiring $g$ where an individual buyer is denoted $i \in \mathbf{N}$. Buyers are interested in acquiring the good/service $g$ because of some intrinsic value they place on it.

**Definition 11** : *A buyer's valuation price, $v_i \forall i \in \boldsymbol{N}$, is known privately by each individual buyer and is the maximum non-negative price he is willing to pay to acquire a single identical unit of an indivisible good/service, $g$.* [72]

Valuations prices are only known to the individual buyer/seller and unknown to all other buyers. The valuation price is the monetary

equivalent to the value a buyer $i$ would gain by winning the auction and obtaining $g$. The vector $\mathbf{v} = (v_1, ..., v_n)$ denotes the valuation prices associated with the buyers. Buyers compete with each other by submitting bids publicly that reflect the intrinsic value they place on the $g$.

**Definition 12** : *A buyer's bid, denoted $b_i(v_i)$, is a report from buyer i that represents the amount they are willing to pay for g as a function of their private valuation price, $v_i$.* [100]

The vector $\mathbf{b}(\mathbf{v}) = (b_1(v_1), ..., b_n(v_n))$ denotes the collection of bids for all participating buyers $i \in N$. To simplify notation a buyer's bid, $b_i(v_i)$, is referred to by $b_i$.

Auction design is the process of eliciting bids, $\mathbf{b}$, with the purpose of determining the final allocation of the good or service $g$ as well as the price to be paid $p$. There are many different formats to accomplish this depending on the setting and desired outcomes. The most familiar auction format described above known as the classic ascending price auction, or English auction and is just one of many common auction formats. Historically, there are four major formats of one-sided auctions.

- **English Auction**:
  The English auction, or ascending auction, is perhaps the most well known auction and is the first of two open-cry auctions [72]. This is classified as an open cry auction because the auctioneer starts by calling out an initial price, and then increases the price by a small amount as bidders openly state, or cry, their interest in acquiring the item. This continues until a single buyer is left willing to pay at the current price. The last remaining buyer is the one with the largest bid, $b_1$, who wins the auction and is allocated the good/service $g$. the price to be paid is equal to the winning buyer's bid, $p_{english} = b_1$. A common example for this type of auction in the modern age is internet auctions like those found on eBay.

- **Dutch Auction**:
  The second open-cry auction is the Dutch auction, otherwise known as the descending auction [72]. Contrary to the ascending auction, here the auctioneer begins at an extreme price and incrementally lowers the price until a bidder claims the good/service at the price mentioned. The winning buyer is the one willing to pay the most, i.e. has the largest bid, $b_1$. Like the English auction price to be paid is equal to the winning buyer's bid, $p_{dutch} = b_1$. The difference in these auctions is the dutch auction will tend to elicit a larger price, $p_{dutch} > p_{english}$. This is because the last remaining bidder in an English is likely to be willing to pay more but does not need to outbid them-self. In a dutch auction bidders do not know what the other bidders are willing to pay. As soon as a price is announced that the largest bidder is willing to pay and they do not wish to risk losing the auction will announce

their bid to end the auction. In this auction it is unlikely the largest bidder is willing to pay much more than their stated bid. This auction type can be found in an Initial Public Offering (IPO) of a company. Alternatively, governments use this auction type for the pricing of financial instruments such as Treasury bills, notes, and bonds.

- **First price Sealed Auction**:
  In contrast to the first two auctions in which bids are made publicly, sealed auctions keep bids hidden. These types of auctions are conducted in phases, the first in which bids are collected, the second in which the winner (if anyone) is determined. The winner proceeds to pay the exact price that was bid. Strategically the first price sealed auction is equivalent to the dutch as they both results in the same equilibrium. Experimentally they two auction formats can produce different results due to the open-cry environment and added suspense of waiting on the clock [27]. First price sealed auction are commonly used in government/military acquisitions.

- **Second price Sealed auction**:
  For a single-item, one-sided auction the second-price auction is perhaps the golden standard taking the number 1 slot if crafting a top 10 auctions list [82].

  One of the earliest documented cases for this auction comes from 1797 where Johann Wolfgang von Goethe sold a manuscript using a sealed-bid, second-price auction. Today this is also referred to as the Vickery Auction, the winner (highest bidder) in this type pays the price of the second highest bidder, $p = b_2$ instead of his own bid, $b_1$.

  When the winners own bid is used in the determination of the price, bidders must be strategic so as to bid enough to win the auction, but not more than necessary to win so as to pay a lower price. Not using the winners bid allows buyers to raise their bids, without impacting price, up until the maximum amount they would be willing to pay for the good/service. Buyers of course are not interested in paying more than what the good/service is worth to them. If they bid is not sufficient to win the auction, the buyer does not pay anything, and if the bid does win, the buyer pays less than or equal to the bid he submitted.

  The sealed bid second price incentives buyers to bid the maximum possible subject only to their true valuation of the good or service as the upper limit. In fact, the second-price auction is an example of game that has a weakly dominant strategy equilibrium. A winning bidder who increases their bid beyond their valuation has no impact on the outcome as they still win the auction and pay the second highest price. A losing bidder who increases their bid beyond their valuation could end up winning the auction and paying more than they value the good

or service for and coming out worse off then if they had lost. A winning bidder who decreases their bid from their valuation runs the risk of losing the auction. losing bidders who decrease their bid have no effect on their own outcome and still loose the auction. As bidders can at best receive the same result if not worse by bidding a strategy different than their true valuation the second price auction has a weakly dominant strategy equilibrium in which all buyers (players) bid their valuation of the good/service. Auctions that incentivize bidders reveal their valuation are called truthful. This is an important property of mechanism design discussed in Section 2.2.2.

This style of auction is seen in specified pool (SP) trading in the agency mortgage-backed securities (MBS) market. The U.S. Treasury Department also experimented with this to sell of the national debt in the 1970's [72].

Naturally the existence of first and second price auctions implies the existence of third and fourth price auctions up to an arbitrary number limited only by the number of auction participants. The range of prices can all be generalized into the $k$th price auction. Where $k = 2$ would equate to the second price auction. The larger $k$ becomes the lower the price will be and the happier the winning buyer becomes. This of course comes at the cost of revenue for the auction itself.

Another approach towards auction design is to repeat the auction many times rather than just conduct the auction in one-shot. In a repeated auction there are several periods of trading wherein buyers and sellers are set up with a new initial allocation of resources. It can be presumed that in between periods goods are produced or consumed by the buyers and sellers. Repeated auctions allow the traders an opportunity to learn and optimize their payoffs according to past auction results [71].

Repeated auctions allow for the traders to learn the behavior and strategy of other traders even if different items that hold different valuations are used. Buyers need not be the only traders participating in an auction. In fact buyers make up only one side of a larger picture. In any commercial setting buyers engage with sellers to make up the most simple version of a market. Double Auctions incorporate both sides of the market, that is buyers and sellers.

All of the above auctions consist in some form of buyers engaging in competitive behavior to secure some good or service and are depicted in the middle graphic in Figure 2.1 [97]. These formats can be inverted in which the auction would consist of sellers engaging in competitive behavior to sell some good or service to a single interested buyer. While still considered a one-sided auction these formats are known as reverse auctions and are seen in the left-most graphic depicted in Figure 2.1.

Many markets comprise both the traditional group of competitive buyers as well as a group of competitive sellers as emblematic in

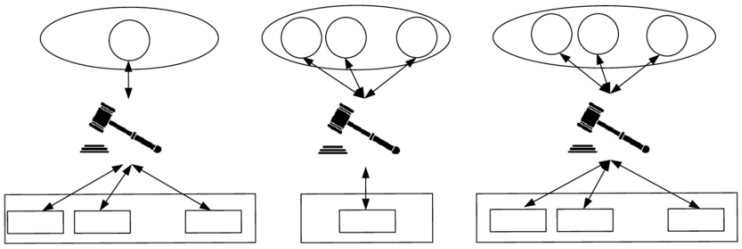


Figure 2.1: Auction Formats

the reverse auction. In these cases a more complex auction format is required that merges both sides of the market. These two-sided auctions are commonly known as Double Auctions and are depicted in the right-most graphic in Figure 2.1. Buyers and sellers both engage in strategic competitive behavior for their chance to be involved in a trade of some good or service.

In addition to a set of buyers **N**, the two-sided auction, known as a double auction (DA), includes a set of sellers **M**. In a one-sided auction the set of sellers would consist of a single trader, or the case of a *reverse* auction the set of buyers would consist of a single trader.

### 2.1.2 *Double (Two-sided) Auctions*

Double auction (DA) institutions consider a set of buyers, **N**, and sellers, **M**, simultaneously. Buyers are uniquely characterized by their private valuation prices as in Definition 11 and submit bids according to Definition 12. Sellers are uniquely characterized by a private reservation price.

**Definition 13** : *A seller's reservation price, $r_j \forall j \in \boldsymbol{M}$, is the minimum non-negative price they are willing to depart with the single identical unit of an indivisible good/service, g, and is known privately by each individual seller.* [72]

Valuations and reservation prices are private, only known to the individual buyer/seller and unknown to all other traders (buyers and sellers). The vector $\mathbf{r} = (r_1, ..., r_m)$ denotes the reservation prices associated with the sellers. Auctions involve traders strategically submitting reports called bids reflecting their valuation/reservation prices.

**Definition 14** : *A seller's bid, denoted $s_j$, is a report from seller j that represents the amount they are willing to sell g for.* [100]

The vector $\mathbf{s} = (s_1, ..., s_m)$ denotes the collection of bids for all sellers $j \in M$.

A seller's bid $s_j$ reported by seller $j$ represents $r_j$. For a seller a bid represents an asking price to part of the good or service in their possession. Strategic behavior allows for $s_j$ to differ from $r_j$. Similarly a bid from buyer $i$ is denoted $b_i$ and represents $v_i$. For a buyer this is

an offer to acquire an asset. Buyers may want to strategically report a bid $b_i$ different than $v_i$.

Auctions, whether one- or two-sided, are implemented with the purpose of simultaneously determining a final allocation of the good/service, $g$, as well as the prices to be paid and/or received. Denote the buyers and sellers assigned to the winning allocation with $\mathbf{T}_b$ and $\mathbf{T}_s$ respectively such that the set of buyers and sellers comprise the entire winning set, $\mathbf{T}_b \dot{\cup} \mathbf{T}_s = \mathbf{T}$ [49].

Following the implementation of the DA, all buyers and sellers are split into two disjoint sets according to the final allocation determination: the winning set, $\mathbf{T} = \mathbf{T}_b \dot{\cup} \mathbf{T}_s$, involved in the exchange of $g$; and the set losing traders, denoted $\overline{\mathbf{T}} = \overline{\mathbf{T}}_b \dot{\cup} \overline{\mathbf{T}}_s$ [49].

All participating traders are either part of the wining set or losing set such that $b_i \in \mathbf{T}_b \dot{\cup} \overline{\mathbf{T}}_b \forall i \in \mathbf{N}$ and $s_j \in \mathbf{T}_s \dot{\cup} \overline{\mathbf{T}}_s \forall j \in \mathbf{M}$ [49].

The submission of bid by a traders acts as a binding agreement to participate in the market. Traders cannot submit bids and then remove themselves from the market to avoid paying. By participating in the auction all buyers will also have a price to be paid and all sellers will have a price to be received.

The price buyer $i$ pays by participating in an auction through the submission of a bid, $b_i$ is denoted $p_{b_i} \in \mathbb{R}$. The vector $\mathbf{p_b} = (p_{b_1}, ..., p_{b_n})$ denotes the collection of prices for all buyers $i \in N$. The price seller $j$ receives by participating in an auction through the submission of a bid, $s_j$ is denoted $p_{s_j} \in \mathbb{R}$. The vector $\mathbf{p_s} = (p_{s_1}, ..., p_{s_m})$ denotes the collection of prices for all buyers $j \in M$ [27].

DA institutions can be delineated between those that run constantly, like the New York Stock Exchange, and those that occur as discrete events with defined starts and finishes for the auction. A constantly running double auction is known as a continuous double auction and is contrasted with the more simplified discrete double auction more commonly known by the name "clearing house". In a continuous-time auction buyers and sellers, known as traders, may engage in acquisitions at any point throughout the defined trading period, and typically participate in many bilateral transaction. Discrete-time auctions only permit traders to move in a single step from initial to final allocation [38].

**Definition 15** : *A double auction is **discrete** if both buyer and seller bids are collected during an initial period, and then cleared at the end of the period deciding the final allocation and payments according to the collected bid messages* [78].

This research limits consideration to **discrete** double auctions by Definition 15. Additionally limit consideration to the bilateral trade case according to Definition 16.

**Definition 16** : *Bilateral trade involves each seller $j \in \boldsymbol{M}$ possessing a single identical unit of of an indivisible good/service $g$ and competing to sell that good/service $g$ to a single buyer $i \in \boldsymbol{N}$* [31].

Bilateral trade DAs do not allow for buyers and sellers to pool their resources in order to acquire and share the good/service $g$.

A discrete bilateral trade DA requires buyers and sellers to post their asks and bids within a given period or length of time. When this period is over, the auctioneer closes the market and constructs demand and supply curves from the pool of bids and asks, identifying how many units would be bought and sold at every price. The market is then cleared by matching buyers and sellers, and a vector of clearing prices is set [76].

Allowing traders to pay their own individualized prices allows an auction to exploit high rollers by charging premium prices simply because they value the good or service more. This is a cause for concern among participating traders due to the differentiation in prices for the same good/service. Traders become envious of other traders getting a better deal.

**Definition 17** : *A discrete bilateral trade double auction is non-discriminatory if all buyers pay (sellers receive) identical payments for the same good/service $g$* [27].

In Non-discriminatory auctions all winning buyers and sellers are treated the same regardless of the bid they submitted. Formally this reduces the vector buyer and sellers prices down to a single value; $\mathbf{p_b} = (p_{b_1}, ..., p_{b_n}) \rightarrow p_{b_i} = p_b \forall i \in \mathbf{N}$ and $\mathbf{p_s} = (p_{s_1}, ..., p_{s_m}) \rightarrow p_{s_j} = p_s \forall j \in \mathbf{M}$.

The implementation of an auction mechanism does not guarantee that a trade must occur. Traders desire to participate in the exchange of a good/service $g$ if it benefits them and improves their utility according to Definitions 4 and 5.

### 2.1.3 *Measures of Performance*

When comparing multiple auction designs as economic institutions there are two main performance areas; revenue and efficiency.

**Definition 18** : *The revenue of a double auction mechanism, X is total amount of money that is produced by selling the goods or services to buyers* [49].

By continuing to limit consideration to auctions that do not charge a participation fee traders assigned to the losing allocation, $\overline{\mathbf{T}}$, do not pay or receive anything. This follows the utility functions defined by equation 2.7a for buyers and equation 2.7b for sellers. By Definition 18 only traders assigned to the winning allocation, $\mathbf{T}$, are considered when calculating the revenue which is denoted as $\Pi$. Revenue is calculated according to equation 2.1.

$$\Pi_X = \sum_{b_i \in \mathbf{T}_b} p_{b_i} - \sum_{s_j \in \mathbf{T}_s} p_{s_j} \tag{2.1}$$

Equation 2.1 can be further reduced in non-discriminatory auctions as all winning buyers pay the same price $p_b$ with all winning sellers receive $p_s$. Denote the number of winning buyers in $\mathbf{T}_b$ as $n^w$ and the number of winning sellers in $\mathbf{T}_s$ as $m^w$.

$$\Pi_X = n^w * p_b - m^w * p_s \tag{2.2}$$

Note that in a bilateral trade setting the number of winning buyers equals the number of winning sellers, $n^w = m^w$. Equation 2.2 can be further simplified by using either $n^w$ or $m^w$.

$$\Pi_X = n^w * (p_b - p_s) \tag{2.3}$$

The revenue of a double auction can be thought of as the amount leftover from the money paid by buyers after giving the sellers their earnings. The revenue incentives the existence of the market itself.

The second measure of auction performance deals with the traders themselves.

**Definition 19** : *An efficient market allocates the good or service g to the traders who value it the most.* [49].

The efficiency of an auction measures the degree to which the good/service is allocated to traders with the largest valuation and reservation prices. An efficient auction may be one where no buyer values $g$ more than the seller's reservation prices and determines that no trade should take place and thereby produce a revenue of zero.

A challenge for a DA is the existence of two sides of the market where prices for buyers to pay, $p_b$, and sellers to receive, $p_s$, must be determined that are acceptable to both the buyers and sellers side of the market. This requires the buyer's valuation, or willingness to pay, to exceed the seller's reservation price, or asking price. A trade that benefits each side is said to be efficient.

**Definition 20** : *An efficient trade results in non-negative utility for both the buyer and seller involved in the exchange of a good or service* [72].

Efficient trades satisfy the following conditions for buyer $i$'s valuation price, $v_i$, and seller $j$'s reservation price, $r_j$ in equations 2.4a and 2.4b.

$$v_i \geq p_b \geq r_j \; \forall \; i \in \mathbf{N}; j \in \mathbf{M} \tag{2.4a}$$

$$v_i \geq p_s \geq r_j \; \forall \; i \in \mathbf{N}; j \in \mathbf{M} \tag{2.4b}$$

A pitfall in the measurement of auction efficiency is the sum of the change in utility across all traders. This generates concern when the goal of the auction is to extract as much revenue from the market as possible because a traders utility is dependent upon the prices paid and received. Designing an auction to extract revenue from the market implies that buyers pay the largest amount possible while sellers receive the smallest. This results in auctions that implement prices as close to the private valuation and reservations as possible. In this case the market would generate a minimal amount of utility for the traders despite potentially processing a large amount of trades with considerable gains coming from these trades. Therefore summing up the traders utilities as a measure of market efficiency would fail to capture any gains that are absorbed by some third party, be that third party an auctioneer, broker, market designer, etc. For this reason, an alternate measure is required to properly capture the total market gains that are generated from trade [27].

**Definition 21** : *The gains from trade (GFT) generated by a mechanism, X, is denoted as $\Omega_X$, and is an economic measure of market efficiency. $\Omega_X$ and is calculated using the buyer's bids as a function of their private valuation prices $\boldsymbol{b}(\boldsymbol{v})$ and seller's bids as a function of their reservation prices $\boldsymbol{s}(\boldsymbol{r})$ according to equation 2.5* [49].

$$\Omega_X = \sum_{b_i \in \mathbf{T}_b} b_i(v_i) - \sum_{s_j \in \mathbf{T}_s} s_j(r_j) \tag{2.5}$$

As the GFT are a measure of market efficiency according to Definition 19, equation 2.5 demonstrates that the gains from trade is independent of price. The efficiency of a market is based on the allocation of the good/service $g$ not on the utility of individual traders. An ideally efficient market can we result in all buyers paying their valuation prices while all sellers receive their reservation prices. In this case all traders will have a utility of zero while the revenue, according to equation 2.1 is maximized.

While the field of auction theory tends to use gains from trade as a measure of efficiency, social theory adapts an alternative measure of market efficiency known as the social welfare.

**Definition 22** : *A auction's social welfare denoted as* $\overline{\Omega_X}$*, is an economic measure of market efficiency and is calculated using the buyer's bids as a function of their private valuation prices* $\boldsymbol{b}(\boldsymbol{v})$ *and sellers bids as a function of their reservation prices* $\boldsymbol{s}(\boldsymbol{r})$ *according to equation 2.6* [49].

$$\overline{\Omega_X} = \sum_{b_i \in \mathbf{T}_b} b_i(v_i) + \sum_{s_j \in \overline{\mathbf{T}_s}} s_j(r_j) \tag{2.6}$$

The social welfare sums the private valuation and reservations prices of those traders in possession of $g$ after the market is cleared. In other words the social welfare sums the buyer valuations of those involved in a trade with the sum of seller reservation prices that are NOT involved in a trade, and thus maintain possession of $g$ as they are not successful in the trade [12].

Both measures of efficiency, GFT and social welfare, are separate measures with different values.

Consider a market saturated with an abundance of sellers and very few buyers. Assume the final allocation from the auction results in one of three efficient trades taking place. In other words, the final allocation results in two buyers who value the good/service more than two of the sellers. By social welfare standards this may not be a concern due to the large number of sellers who maintain possession that are included in the calculation of the measure. Comparing to the optimal allocation may even result in relatively high efficiency.

The GFT does not dilute the measure in seller saturated markets and is only concerned with the efficient trades. Comparing the GFT when only one of three trades goes through with the **optimal allocation** would reveal a much measure of efficiency that highlights the failure to realize efficient trades. For this reason this work, and auction theory at large use GFT as a measure of performance for an auctions ability to clear the market efficiently.

### 2.1.4 *Example*

Consider a three buyer-three seller auction with valuation/reservation prices and strategic bid determinations according to Table 2.1.

Buyer and Seller bids example

| Private reservation price | Seller bids | Buyer bids | Private valuation price |
|---|---|---|---|
| 15 | 15 | 21 | 23 |
| 12 | 14 | 20 | 25 |
| 7 | 10 | 7 | 16 |

Table 2.1: Example: bids

An auction can be designed that determined a single trade to take place where the price buyer's pay is $p_b = 20$ and the price the seller's receive is $p_s = 14$. Utility for each player can then be calculated according to their private valuation and reservation prices.

Utility example

| Trader | Bid | Price | Private valuation/ reservation prices | Utility |
|---|---|---|---|---|
| Buyer | 21 | $p_b = 20$ | 23 | $23 - 20 = 3$ |
| Seller | 10 | $p_s = 14$ | 7 | $14 - 7 = 7$ |

Table 2.2: Example: utility

In this example the seller generated significantly more utility then the buyer. Often auctions are designed such that the host receives compensation for their effort, meaning the price a winning buyer pays $p_b$ can be different from what the seller receives $p_s$. The revenue is equal to the price the buyers pay subtracted by the price the sellers receive, $\Pi = 20 - 14 = 6$.

The gains from trade can also be calculated. The winning buyer has a valuation of 23 while the winning seller has a reservation price of 7. The GFT, $\Omega$, in this example would be $23 - 7 = 16$. Notice that GFT in this example are not optimized as a losing trader exists with a larger valuation 25 that would increase the total GFT to $25 - 7 = 18$. For comparison the related measure of efficiency, the social welfare, is equal to the highest buyer to win the auction 23 added with the two sellers who maintain possession of the good/service 15, 12 for a total of 50.

## 2.2 MECHANISM DESIGN

Auctions are used to find a solution to two questions simultaneously: Who wins the auction, and how much shall they pay/receive? The answer to each of these questions depends upon the other, who trades depends on how much should be paid, and how much should be paid depends on who is involved in the trade. These questions can be answered through two interrelated rules.

**Definition 23** : *An allocation rule is a function that takes in the vectors of buyer and seller bids and returns the set of winning and losing bids,* $A : (\boldsymbol{b}, \boldsymbol{s}) \rightarrow \boldsymbol{T} \dot{\cup} \overline{\boldsymbol{T}}$ [27].

**Definition 24** : *A payment rule is a function that takes in the vectors of bids* $\boldsymbol{b}, \boldsymbol{s}$ *and returns a tuple of price vectors for buyers and sellers,* $P : (\boldsymbol{b}, \boldsymbol{s}) \rightarrow \boldsymbol{p} = (\boldsymbol{p_b}, \boldsymbol{p_s})$ [27].

These rules work together to determine the winners of the auction and the prices to be paid/received. Combining these two rules create a mechanism, by Definition 25, which acts as a function of functions that takes in trader bids and applies the allocation and payment rules.

**Definition 25** : *A mechanism X defines the collection of allocation and payment rules,* $X(A, P) : (\boldsymbol{b}, \boldsymbol{s}) \rightarrow [\boldsymbol{T}\dot{\cup}\overline{\boldsymbol{T}}, \boldsymbol{p}]$ [27].

Mechanisms exists to allocate goods/services at specific prices to individual bidders as characterized by the allocation and payment rules in Definitions 23 and 24.

A traders utility function can then be determined according to four things: 1. a traders private valuation/reservation price, $v_i/r_j$; 2. a traders buyer/seller bid, $b_i/s_j$ price; 3. the final allocation according to the allocation rule, $A : (\mathbf{b}, \mathbf{s}) \rightarrow \mathbf{T}\dot{\cup}\overline{\mathbf{T}}$; and 4. the price to be paid/received, $P : (\mathbf{b}, \mathbf{s}) \rightarrow \mathbf{p} = (\mathbf{p_b}, \mathbf{p_s})$.

Limiting consideration to non-discriminatory auctions by Definition 17 that do not charge a participation fee allows the formulation of quasi-linear utility functions according to equation 2.7a for buyers and equation 2.7b for sellers.

$$u_i(v_i, X(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow [\mathbf{T}\dot{\cup}\overline{\mathbf{T}}, (p_b, p_s)]) = \begin{cases} 0 & b_i \in \overline{\mathbf{T}}_b \\ v_i - p_b & b_i \in \mathbf{T}_b \end{cases} \tag{2.7a}$$

$$u_j(r_j, X(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow [\mathbf{T}\dot{\cup}\overline{\mathbf{T}}, (p_b, p_s)]) = \begin{cases} 0 & s_j \in \overline{\mathbf{T}}_s \\ p_s - r_j & s_j \in \mathbf{T}_s \end{cases} \tag{2.7b}$$

If a buyer or seller looses the auction, the traders are unaffected and maintain a utility of zero. Charging a participation fee could result in negative utilities for losing traders. Winning the auction yields a positive utility so long as buyers pay less ($v_i > p_b$), and sellers receive more ($p_s > r_j$),than what the good/service is worth to them. Buyers and sellers thus engage in strategic behavior to influence the prices to be paid/received.

**Definition 26** : *Buyers (Sellers) are strategic utility maximizers that select bids according to their private valuation (reservation) prices and the mechanism designed for the auction.*

When designing market-clearing mechanisms, four key economic properties must be considered: Individual Rationality (IR), which incentivizes market participation; Incentive Compatibility (IC), which encourages traders to truthfully reveal private information; Perfect Efficiency, which ensures all trades with net positive gains are processed; and Budget Balance (BB), to incentivize the existence of the market itself. This section examines these four properties in detail.

For the sake of readability mechanism notation is often slightly reduced, at no cost in detail, to only show the winning allocation $X : (\mathbf{b}, \mathbf{s}) \rightarrow [\mathbf{T}, \mathbf{p}]$.

### 2.2.1 *Individual Rationality (IR)*

When designing an auction, it's crucial to incentivize traders to participate. By Definition 26, traders are strategic utility maximizers. To encourage participation, the auction must guarantee non-negative utility. This way, participants are no worse off than if they had not participated, while having the potential to increase their utility.

**Definition 27** : *A trader is defined to be Individually Rational (IR) if he has a non-negative utility.*

$$v_i \geq p_b \; \forall \; b_i(v_i) \in \mathbf{T}_b \tag{2.8a}$$

$$r_j \leq p_s \; \forall \; s_j(r_j) \in \mathbf{T}_s \tag{2.8b}$$

This definition introduces three different version of IR:

- *Ex-ante IR*:
  Traders know only the value/reservation price distributions and have not observed their own valuation/reservation price. This is appropriate for many long term and forward contracts, where contracting occurs that bind traders before they are able to realize their private price. An example of this is in maintenance/repair, a seller of services may need to accept a contract before realizing the actual workload or unexpected difficulties of a project. Similarly a buyer may need to commit to a purchase before understanding the effect the good/service will have. They buyer may not know their true valuation until after the purchase when it can be put to use. Ex-ante IR is based on expectation of utilities given the known distributions [27].
- *Ex-interim*:
  Each trader knows their own private valuation or reservation price. IR is based on the expected price to be paid should the trader win the auction. Utility is still **expected** to be non-negative under this condition [36].
- *Ex-post*:
  Traders know both their valuation/reservation prices as well as the final payment. Under this most strict condition utility is always non-negative [36].

*Ex-ante* IR is sufficient in auctions where private valuations follow a publicly known distribution function. *Ex-interim* IR is useful in non-deterministic auctions in which traders must make bids on expectation, however the auction design may not guarantee the bid be IR after the market is cleared. An example of this can seen in a market where seller $j$ expects to auction of some $g$ that is seen as a collectable and thus warrants valuations that are expected to exceed $r_j$. If buyers are not collectors and are unaware of the value one might find that bids are significantly lower than expected and the seller receives a payment below what they actually were comfortable selling it for. Finally, the strongest of these notions *ex-post* IR is met even after the auction concludes. Unless stated otherwise auctions that achieve the desired property of IR do so ex-post. Consider the example in Table 2.1 where the all traders clear the market at a single price $p = p_b = p_s = 14$ shown in Table 2.3.

IR example

| Private reservation price | Seller bid | Price | Buyer bid | Private valuation price |
|---|---|---|---|---|
| 15 | 15 | 14 | 21 | 23 |
| 12 | 14 | 14 | 20 | 25 |
| 7 | 10 | 14 | 7 | 12 |

Table 2.3: Example: Individual Rationality (IR)

In this example the least competitive buyer submitted a bid report stating they would be willing to pay 7 and won the auction at a price of 14 rather than simply not trading. With a private valuation price less than the price to be paid this buyer will end up with a negative utility thereby violating the desired property of IR.

As valuation and reservation prices are private information unknown to the auction designer achieving IR in non-truthful auctions is accomplished according to the bids rather than the unknown private valuation/reservation prices. In these auctions it is assumed that a buyer $i$ will only submit a bid that is less than their private valuation price, $b_i \leq v_i$, and a seller $j$ only submits a bid that exceed their private reservation price, $s_j \geq r_j$. This allows the auction to only allow trades where the price buyers pay is lower than the stated bids for winning buyers, $p_b \leq b_i$, and exceeds the bids for winning sellers, $p_s \geq s_j$.

Bids that differ from private valuation/reservation prices can lead to inefficiency. The possibility exists that a seller's bid exceeds buyer's bid, $s_j > b_i$ indicating an efficient trade does not exist when in fact the private valuation prices might prove it does, $r_j < v_i$. This problem can be alleviated through careful design to incentives traders to truthfully reveal their private information via their bids. This desired economic property of truthfulness is known as incentive compatibility.

### 2.2.2 *Incentive Compatibility (IC)*

Classic economic theory conceptualizes the existence of an "invisible hand" that leads markets to end in efficient resource allocation accounting for all necessary variables, to include traders seeking to maximize their own utility. This paradigm faces difficulties with the introduction of private information. Information that the "invisible hand" cannot and does not account for that leads to market failure. Efficient trades desirable for all parties may fail to occur in the presence of asymmetric information [14].

In mechanism design this asymmetric information is the communication and information flow between traders. Acquiring information that may bound private valuations with a certain amount of confidence provides a competitive advantage. Effective negotiation strategies lead to larger payoffs and can draw a significant number of resources to conduct complex strategic analysis. Fortunately, a mechanism designer can carefully construct allocation and payment rules to simplify the trader's strategic problem immensely.

In an auction the strategic behavior undertaken by traders is represented via bids that reflect their private valuations. Traders wish to bid competitively enough to "win" the auction and join the trading allocation, while also optimizing the price to create the largest payoff. This strategic problem dissipates when the traders optimal bid is their true valuation. In other words, of all possible bids a trader can make, if the bid that results in the highest expected payoff independent of all other bids is the bid that truthfully reveals their private information; then the strategic problem is reduced from determining an optimal bid to reflect their true valuation, to the bid that is exactly equal to their private information [49].

Auctions designed in this way are said to be strategy proof, an example of this from the one-sided auctions introduced previously is the second-price auction in which the highest bid wins the auction but pays the bid of the second highest price. This incentives the bidders to not undercut their true valuations as the price is determined independent of the bid. Buyers are incentivized to report the largest possible bid limited only by the maximum amount they are willing to pay to be indifferent between winning the auction and receiving the good or service and losing the auction but keeping their money. This limit is equal to their private valuation price. This auction thus incentives bidders to truthfully reveal their valuation price with their bid as this strategy maximizes their expected payoff regardless of the behavior of the other bidders. This allows for the following definitions.

**Definition 28** : *A **Dominant Strategy Bid** is one that maximizes a bidder's utility, regardless of what other bidders do* [82].

This allows for the definition of the truthfulness property for auction mechanism design: Incentive Compatibility (IC).

**Definition 29** : *An auction is **Incentive Compatible (IC)** if truthful bidding is always a dominant strategy for every bidder* [82].

Consider a second-price auction where every buyer $i \in \mathbf{N}$ has a dominant strategy to report $b_i = v_i$. If the vector of all bids is denoted as $\mathbf{b}$ then the vector of bids excluding $b_i$ can be denoted $\mathbf{b}_{-i}$. Define $b_{max} \in \mathbf{b}_{-i}$ as the largest of all other bids. Of all possible bids buyer $i$ can make only two outcomes can occur. Either $b_i < b_{max}$ in which case $i$ looses and obtains a utility of 0, or $b_i \geq b_{max}$ in which case he wins and pays $b_{max}$ and obtains a utility $v_i - b_{max}$. If $v_i < b_{max}$ winning would result in a negative utility. Thus, regardless of $b_i$ the maximum utility buyer $i$ can obtain is 0. This maximum can be achieved by simply reporting $b_i = v_i$. Alternatively, if $v_i \geq b_{max}$ the maximum utility $i$ can obtain is $v_i - b_{max}$, which is achieved by reporting $b_i = v_i$.

Additionally second-price auctions ensures that truthful bidders are guaranteed non-negative utilities. As losing traders receive a utility of 0, it is sufficient to prove that winners receive a utility $\geq 0$. If buyer $i$ wins, he receives a utility of $v_i - p_b$, where $p_b$ is defined by the second highest bidder $b_{max}$. Winning requires $b_i \geq b_{max}$. As $b_i = v_i$ and $b_{max} = p_b$ then $v_i \geq p_b$ and $v_i - p_b \geq 0$.

The main benefit from IC auctions is the simplicity for participation. Strategic behavior is removed as it is in the buyers best interest to truthfully reveal their valuation price. Pairing this with the Individual Rationality (IR) property introduced in Section 2.2.1 ensures participation in the market itself is incentivized through the non-negativity guarantee.

### 2.2.3 *Revelation Principle*

Problems presented in mechanism design literature can be simplified through the reliance on the revelation principle which states that the outcome of any mechanism $X$ can be replicated by a new direct mechanism, $\overline{X}$, that simulates the equilibrium strategies of the traders.

**Definition 30** : *In a direct mechanism traders report their bids in an initial stage, after which all bids are collected and the market is cleared according to the rules defined by mechanism $X$.* [69].

Figure 2.2 depicts this relationship graphically where the direct revelation mechanism takes in the private valuation and reservation prices and simulates the auction mechanism by running each traders dominant strategy in equilibrium on their behalf. Examples of direct mechanisms are first-, second-, and third price auctions or even an all-pay auction.

In contrast to direct mechanisms, in an indirect mechanism traders to not report their bid, rather the auction may iteratively suggest prices for traders to interact with. Examples of indirect mechanisms are the English, or ascending, auction, and the Dutch, or descending auction introduced in Section 2.1.1.

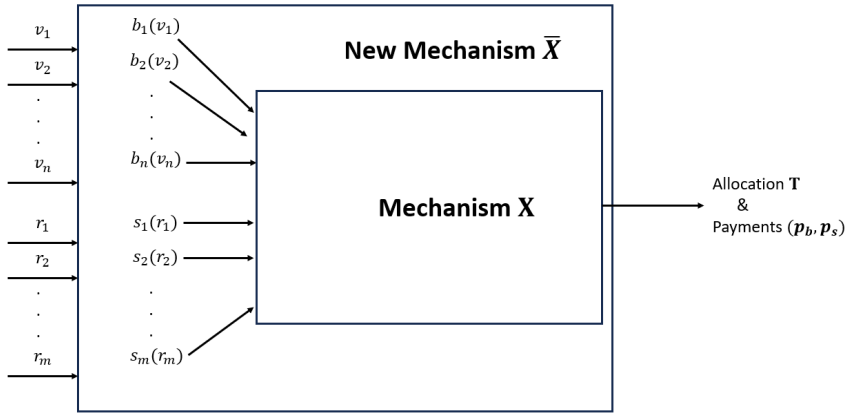



Figure 2.2: Direct Revelation Mechanism Relationship

Formally, let $\mathbf{b} = b_1(v_1), ..., b_n(v_n)$ be the dominant strategy in equilibrium for buyers and $\mathbf{s} = s_1(r_1), ..., s_m(r_m)$ for sellers of the original mechanism $X(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow [\mathbf{T}, \mathbf{p}]$. Define a new direct revelation mechanism: $\overline{X}(A, P) : (\mathbf{v}, \mathbf{r}) \rightarrow [\mathbf{T}, \overline{\mathbf{p}} = \mathbf{p}]]$ such that buyers submit their private valuation and reservation prices and the new mechanism produces the same allocation and payments to the original.

**Revelation Principle**: *When every trader in a mechanism X has either a weakly or strictly dominant strategy according to Definition 7, there exists an equivalent direct revelation dominant strategy IC mechanism $\overline{X}$ that can be constructed.* [39]

*Proof.* Dominant strategy buyer bids excluding buyer $i$ are denoted $\mathbf{b}_{-i}$, and dominant strategy seller bids excluding seller $j$ are denoted $\mathbf{s}_{-j}$. Given each bid $b_i(s_j)$ is a dominant strategies for buyer $i$ (seller $j$), then every alternate bid that each buyer (seller) could submit,$\hat{b}_i(v_i), \mathbf{b}_{-i}(\hat{s}_j(r_j), \mathbf{s}_{-j})$, would result in a utility that is less than or equal to the utility they would achieve using their dominant strategy by Definition 6 seen in equation 2.9a for buyers and equation 2.9b for sellers.

$$u_i(v_i, X(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow [\mathbf{T} \dot{\cup} \overline{\mathbf{T}}, (p_b, p_s)]) \geq u_i(v_i, X(A, P) : (\mathbf{b}_{-i}, \hat{b}_i, \mathbf{s}) \rightarrow [\mathbf{T} \dot{\cup} \overline{\mathbf{T}}, (p_b, p_s)]) \quad (2.9a)$$

$$u_j(r_j, X(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow [\mathbf{T} \dot{\cup} \overline{\mathbf{T}}, (p_b, p_s)]) \geq u_j(r_j, X(A, P) : (\mathbf{b}, \mathbf{s}_{-j}, \hat{s}_j) \rightarrow [\mathbf{T} \dot{\cup} \overline{\mathbf{T}}, (p_b, p_s)]) \quad (2.9b)$$

This produces the definition of incentive compatibility of the new mechanism $\overline{X}(A, P) : (\mathbf{v}, \mathbf{r}) \rightarrow [\mathbf{T}, \overline{\mathbf{p}}]$ by Definition 29 [82]. □

Traders are incentivized to report their true private valuations, because by doing so, the direct revelation mechanism $\overline{X}$ produces the same outcome as $X$ which by assumption is in dominant strategy equilibrium [69]. Lying would leave any one trader worse off.

The revelation principle makes it sufficient to restrict attention to direct mechanisms without loss of generality as it can be applied to all mechanisms to convert them into direct Incentive Compatible (IC) mechanisms [69]. This means that if the desire is to design a mechanism that has dominant strategies, might as well make it direct revelation.

The difficulty in designing IC auctions is the requirement that an outcome is a dominant-strategy equilibrium by Definition 8 [82].

### 2.2.4 *Ideal Efficiency*

An efficient trade by Definition 20 is one that it is mutually beneficial to both traders in a bilateral trade. Recall from Section 2.1.3 that efficiency is a measure of performance for a market institution like an auction mechanism. The gains from trade by Definition 21 serve as a way to quantify the efficiency of a mechanism. This value by itself can provide little context as the relative efficiency a mechanism could achieve. For this a third desired economic property that a mechanism can be designed to achieve is introduced, that of ideal efficiency.

This property ensures that all the possible efficient trades are realized thereby optimizing the gains from trade (GFT) as a measure of market efficiency. This allows for the comparison of mechanisms to provide context into the relative efficiency a mechanism has achieved. Truthful mechanisms, or dominant strategy incentive compatible (IC) mechanisms, allow for the ideally efficient allocation to be determined as well as the calculation of the optimal GFT the market can achieve.

Equations 2.10 and 2.11 organize bids according to their competitiveness, as in the McAfee Trade Reduction mechanism [60], where ties are further broken lexicographical by ID such that the first and most competitive buyer is the buyer that submits the largest bid.

Likewise the most competitive seller is the seller that is willing to part with the good/service for the lowest possible amount, or smallest bid.

$$b_1 \geq b_2 \geq ... \geq b_n \tag{2.10}$$

and

$$s_1 \leq s_1 \leq ... \leq s_m \tag{2.11}$$

The organization of bids this way creates what is known as order statistics.

**Definition 31** : *Order statistics are values of sample observations sorted and indexed in either ascending or descending order* [30].

Organizing bids according to their competitiveness is common practice in mechanism design as it allows for the determination of the number of efficient trades. Assuming bilateral trade under Definition 16 the number of efficient buyer bids must be equal to the number of efficient seller bids. By equations 2.12 and 2.13 there exists some number of efficient buyer and seller bids, denoted by $k$, such that at trade is mutually beneficial between buyer and seller $k$ but not between buyer and seller $k+1$.

$$b_k \geq s_k \tag{2.12}$$

and

$$b_{k+1} < s_{k+1} \tag{2.13}$$

Regardless of which buyer $b_1$ through $b_k$ trade with which seller $s_1$ through $s_k$, the use of order statistics allows for the determination of where the line is to be drawn between efficient and inefficient trade.

The efficient number of traders, $k \leq \min\{m, n\}$, are defined by the allocation rule. Consider the special case when $k = n$, that is all of the buyers in the market are efficient and buyer bid $b_{k+1}$ does not exist. To deal with this McAfee introduces fictitious bid that is sufficiently small to take the place of $b_{n+1}$ [60]. By normalizing all bids on a scale of $[0, 1]$ the fictitious buyer bid $b_{n+1}$ would act as the lower bound and have a value of zero. In this case sellers $s_1$ to $s_n$ are also efficient. Similarly if $k = m$, all sellers are efficient which would require the use of a sufficiently large fictitious seller bid $s_{m+1}$. On a normalized scale $s_{m+1}$ would acts as the upper bound and have a value of one. In this case the $m$ most competitive buyers are efficient. When $k = m = n$ all traders in the market are efficient.

The pair of traders $(b_k, s_k)$ are efficient traders, but are the least efficient of all efficient traders. This pair is known as the least favorable trade (LFT). It is not assumed that buyer $k$ trades with seller $k$, rather any efficient buyer can trade with any efficient seller.

**Definition 32** : *The ideally efficient set of traders is denoted as* $\mathbf{\Gamma}$ *and is made up of the ordered buyers and sellers 1 through k.*

Any allocation $\mathbf{T} \neq \mathbf{\Gamma}$ that does not include all $k$ efficient buyers and sellers is sub-optimal, or non-ideal. The optimal allocation, in which $\mathbf{T} = \mathbf{\Gamma}$, yields the ideal GFT shown in equation 2.14:

$$\Omega_{\mathbf{\Gamma}} = \sum_{b_i \in \mathbf{\Gamma}_b} b_i - \sum_{s_j \in \mathbf{\Gamma}_s} s_j \tag{2.14}$$

Mechanism that are designed to hold the desired property of IC ensure the bids truthfully reveal the traders private valuation and reservation prices, $b_i = v_i \forall i \in \mathbf{N}$ and $s_j = r_j \forall j \in \mathbf{M}$. In IC mechanisms equation 2.14 is known to be the optimal market allocation.

**Definition 33** : *The set of inefficient traders is denoted as* $\overline{\mathbf{\Gamma}}$ *and is made up of the ordered buyers* $k+1$ *through* $n$ *and sellers* $k+1$ *through* $m$.

All traders are accounted for and determined to fall into either the efficient or inefficient set, $\mathbf{\Gamma} \dot{\cup} \overline{\mathbf{\Gamma}} = \mathbf{N} \dot{\cup} \mathbf{M}$. Where the set of traders is made up of buyers and sellers $\mathbf{\Gamma} = \mathbf{\Gamma}_b \dot{\cup} \mathbf{\Gamma}_s$ and $\overline{\mathbf{\Gamma}} = \overline{\mathbf{\Gamma}}_b \dot{\cup} \overline{\mathbf{\Gamma}}_s$.

**Definition 34** : *An ideally efficient mechanism* $X^*(A, P)$ *is one where* $A : (\boldsymbol{b}, \boldsymbol{s}) \rightarrow \mathbf{\Gamma} \dot{\cup} \overline{\mathbf{\Gamma}}$ [49].

Note that the measure of efficiency is exclusively concerned with the allocation rule. The price plays no roll.

Consider another allocation rule from an inefficient mechanism $\overline{X}$ that achieves a GFT of $\Omega_{\overline{X}}$ equal to some fraction of the ideal efficiency, $\Omega_{X^*}$. Equation 2.15 measures the efficiency of a mechanism relative to ideal efficiency.

$$e = \frac{\Omega_{\overline{X}}}{\Omega_{X^*}} \tag{2.15}$$

An ideal, or perfectly efficient mechanism has $e = 1.0$. Equation 2.15 makes it possible to compare the relative efficiency of multiple inefficient mechanisms and gives mechanism designers the ability to set desired efficiency thresholds they wish a mechanism to achieve.

### 2.2.5 *Budget Balance (BB): The VCG Mechanism*

Perhaps the most well studied mechanism is the Vickrey-Clarke-Groves (VCG) mechanism, named after their individual contributions [94], [28], and [42]. The initial design for one-sided markets generalized the second-price auction. The extension to the double auction setting was first published by Yoon [101].

The VCG mechanism carries the desirable economic properties of IR, IC, and is ideally efficient. The key concept behind the mechanism lies in the payment rule. Winning traders are charged according to the effect their participation in the market has on the resulting gains from trade seen in equations 2.16a and 2.16b.

$$p_b^i = b_i - (\Omega_X - \Omega_{X_{-i}}) \quad \forall b_i \in \mathbf{T}_b \tag{2.16a}$$

$$p_s^j = s_j + (\Omega_X - \Omega_{X_{-j}}) \quad \forall s_j \in \mathbf{T}_s \tag{2.16b}$$

In the case where the VCG mechanism does not result in any efficient trades, $k = 0$, there are no prices to be made by buyers, or received by sellers.

As the VCG Double Auction (DA) mechanism $X_{VCG}(A, P)$ is IC, the allocation rule, $A$, will result is all $k$ efficient traders assigned to the winning allocation. In the bilateral trade setting, removing a single trader can have one of two effects: 1. The next most competitive trader replaces the removed one in the set of efficient traders, 2. The number of efficient traders are reduced by one from $k$ to $k - 1$.

The gains from trade attributed to the efficient $k - 1$ trades remain unaffected when removing a single trader from the market and cancel each other out when subtracting the gains from trade. Additionally, as prices are only calculated for efficient traders, the removed traders bid is included in $\Omega_X$ and not in $\Omega_{X_{-i}}/\Omega_{X_{-j}}$ and thus cancels out as well.

Equations 2.16a and 2.16b can then be reduced to the least efficient traders $k$ and the most competitive inefficient traders $k + 1$ as they may replace the removed trader. This also results in a single uniform price for each side of the market, shown in equations 2.17 and 2.18.

$$p_b = \max(s_k, b_{k+1}) \tag{2.17}$$

$$p_s = \min(s_{k+1}, b_k) \tag{2.18}$$

The resulting VCG mechanism can then be formalized according to equation 2.19.

$$X_{VCG} : (\mathbf{b}, \mathbf{s}) \rightarrow (\mathbf{\Gamma}, \{\max(s_k, b_{k+1}), \min(s_{k+1}, b_k)\}) \tag{2.19}$$

While guaranteed maintain the economic properties of IR, IC, and produce the ideally efficient outcome; the downside of the VCG mechanism is that is produces a negative revenue, $\Pi$, as defined by equation 2.3: $\Pi_{X_{VCG}} = n^w * (p_b - p_s)$.

Using the order statistics relationships defined in 2.13 and 2.12 it can clearly be shown that $p_b \leq p_s$ causing the VCG DA mechanism to run a budget deficit. This would require the market maker to subsidize the market in order to achieve all the other desirable properties in a Double Auction setting. This can be a problem in many situations and leads to the definition of a fourth desirable property in mechanism design.

**Definition 35** : *A mechanism is* ***budget balance (BB)*** *when the revenue produced is non-negative,* $\Pi \geq 0$ [49].

In a non-discriminatory, bilateral trade, discrete double auction mechanism all winning buyers pay $p_b$ and all winning sellers receive $p_s$.

A mechanism can thus be designed to achieve BB by guaranteeing $p_b \geq p_s$. While the IR property incentives traders to participate in the market, the BB property incentives the existence of the market itself.

Applying the VCG mechanism to the ongoing example in Table 2.1 results in trader's being incentived to truthfully reveal their private information. These bids can be organized into order statistics as in equations 2.10 and 2.11 and is shown in Table 2.4.

VCG example

| Competitively ranked trader | Seller bid/ reservation price | Seller's price $p_s$ | Buyer's price $p_b$ | Buyer bid/ valuation price |
|---|---|---|---|---|
| 1 | 7 | 16 | 15 | 25 |
| 2 | 12 | 16 | 15 | 23 |
| 3 | 15 | 16 | 15 | 16 |

Table 2.4: Example: VCG prices

While all three efficient trades are realized and the optimal GFT is achieved, $p_b \leq p_s$ causing more money to be paid to sellers than is being brought in by the buyers, thus requiring a subsidy in order to run the mechanism. There is no incentive for the market to exist outside of a subsidy as the market runs a deficit.

Budget balanced, $p_b \geq p_s$, is an economic property that results in a positive revenue stream for the market maker. Auctions that produce revenue therefore require an entity to absorb it, which may not always be the case. For governments and non-profits, absorbing a revenue can be a deterrent, especially when it comes at the cost of traders. For these types of markets a more strict requirement of BB can used.

**Definition 36** : *A mechanism is* ***strongly budget balance (SBB)*** *when the revenue produced is exactly zero,* $\Pi = 0$ [49].

It is no longer sufficient to require revenue $\Pi \geq 0$ rather revenue must be strictly $\Pi = 0$ and is easiest achieved with a single-uniform price implying $p_b = p_s$.

A benefit offered from strengthening the desired economic property of budget balance (BB) where the revenue is $\Pi \geq 0$ to *strong* budget balance (SBB) where revenue $\Pi = 0$ is the absorption of gains from trade (GFT) as well as a necessity to absorb the excess revenue. A BB requires the central auctioneer or market maker to absorb all revenue, while this may be desired in many cases it can also lead to unhappy traders as the mechanism is seen as unfair leaving them to search for an alternate. An SBB mechanism does not require a central auctioneer and can implement a single price that can be used by both sides of the market, $p_s = p_b$.

### 2.2.6 *Revenue Equivalence Principle*

The introduction of the revenue equivalence principle first requires the following prerequisite definitions.

**Definition 37** : *A mechanism is **normalized** if losing traders $b_i, s_j \in \overline{\boldsymbol{T}}$ have a payment (and utility) of zero, $p_b = p_s = 0$.*

**Definition 38** : *An allocation rule, $A$, is **bid monotonic** if for any buyer $i$ [seller $j$] and any two possible bids, $\hat{b}_i > b_i[\hat{s}_j < s_j]$, if $b_i \in \boldsymbol{T}_b[s_j \in \boldsymbol{T}_s]$ is in the winning allocation according to allocation rule $A : (b_i, \boldsymbol{b}_{-i}, \boldsymbol{s}) \rightarrow \boldsymbol{T}_b \dot{\cup} \boldsymbol{T}_s$ then $\hat{b}_i \in \hat{\boldsymbol{T}}_b[\hat{s}_j \in \hat{\boldsymbol{T}}_s]$ is also in the winning allocation according to the same allocation rule $A : (\hat{b}_i, \boldsymbol{b}_{-i}, \boldsymbol{s}) \rightarrow \hat{\boldsymbol{T}}_b \dot{\cup} \hat{\boldsymbol{T}}_s[A : (\boldsymbol{b}, \hat{s}_j, \boldsymbol{s}_{-j}) \rightarrow \hat{\boldsymbol{T}}_b \dot{\cup} \hat{\boldsymbol{T}}_s]$.*

Mechanisms that can be characterized according to Definitions 37 and 38 allow for an important finding in mechanism design: the revenue equivalence principle, which outlines the conditions in which two auction formats yield the same expected revenue. Applying this to double auctions means that the equilibrium prices paid/received by winning traders in one auction is equal to the equilibrium expected payments for winning traders by the other.

**Revenue Equivalence Principle:** *Assume all buyer's private valuations are independently and identically distributed (i.i.d) according to distribution $F$. Then any normalized mechanism $X$ with a bid monotonic allocation rule $A$ and buyer bids $\boldsymbol{b}$ yields the same expected revenue to the seller* [27].

The proof uses the following concepts from economics.

**Definition 39** : *Marginal benefit is the value a trader places on attaining an additional unit of the good or service, $g$.* [45].

**Definition 40** : *Marginal cost is incremental payment that occurs when exchanging an additional unit of the good or service, $g$.* [45].

*Proof.* Consider a mechanism $X$ with buyer bids $\mathbf{b}(\mathbf{v})$ that represent each buyer's $i$ private valuation $v_i \forall i \in N$. Let $p_b(v_i)$ be the expected payment of buyer $i$ given their valuation $v_i$.

Assume that every buyer $i$ independently draws their valuation $v_i$ from the same cumulative distribution function $F(v)$ with an associated density function $f(v)$ such that $F(\underline{v}) = 0$ and $F(\overline{v}) = 1$. Where $\underline{v}$ denotes the lower bound for all potential valuation prices and $\overline{v}$ the upper bound.

Consider bidder $i$ chooses to bid according to some other strategy not equal to their private valuation price. Denote the bid $z_i \neq v_i$. Given a bid monotonic allocation rule according to Definition 38, bidder $i$ wins if they are the highest bid which occurs with probability:

$$\underbrace{F(z) \times ... \times F(z)}_{N-1\text{times}} = F(z)^{N-1} \tag{2.20}$$

Buyer $i$'s expected utility then becomes:

$$u(i) = \begin{cases} F(z)^{N-1}v_i - p_b(z_i) & b_i \in \mathbf{T}_b \\ F(z)^{N-1}v_i - 0 & b_i \in \overline{\mathbf{T}_b} \end{cases} \tag{2.21}$$

For simplicity denote $G(z) \equiv F(z)^{N-1}$ with associated density function $g(z)$. If buyer $i$ wins the auction with probability $G(z)$ they receive their private valuation $v_i$ and must pay price $p_b(z_i)$ which is dependent upon the order of $z_i$ relative to all other bids but is independent of buyer $i$'s private valuation $v_i$.

Maximizing buyer $i$'s utility for some other bidding strategy, $z_i$, must satisfy the first-order condition seen in equation 2.22 obtained by differentiating buyer $i$'s utility function with respect to $z_i$ and setting it equal to zero.

$$\frac{\partial}{\partial z_i}u(i) = g(z)(v_i) - \frac{\partial}{\partial z_i}p_b(z_i) = 0 \tag{2.22}$$

From equation 2.22, the term $g(z)(v_i)$ represents buyer $i$'s the positive value bidder $i$ receives according to their private valuation by winning the allocation. In economic terms this is marginal benefit in terms of higher probability of winning the auction. The term $\frac{\partial}{\partial z_i}p_b(z_i)$ represents rate of payment buyer $i$ will surrender reflecting their marginal cost.

Rearranging equation 2.22 as shown in equation 2.23 shows bidder $i$ maximizing bidding strategy is to increase their bid $z_i$ until the marginal benefit in terms of higher probability of winning the auction coincides with their marginal cost.

$$g(z)(v_i) = \frac{\partial}{\partial z_i}p_b(z_i) \tag{2.23}$$

Accordingly, any bid $z_i$ less than the largest possible value bidder $i$ is willing to pay $v_i$ is sub-optimal. Therefore, it must be optimal to bid such that $z_i = v_i$. Substituting $v_i$ back into the first order condition from equation 2.22 is shown in equation 2.24.

$$g(v_i)(v_i) = \frac{\partial p_b(v_i)}{\partial v_i} \tag{2.24}$$

Integrating the above equality leads to an expression for the price to be paid to the seller, $p_b(v_i)$. As the objective is to find an expression for the payment to the seller the integration must be calculated for any possible valuation $y$.

$$p_b(v_i) = p_b(0) + \int_0^{v_i} yg(y)\partial y \tag{2.25}$$

Given the mechanism is normalized any buyer $i$ with a valuation of zero $v_i = 0$ will also pay zero, $p_b(0) = 0$. This reduces the above equality to:

$$p_b(v_i) = \int_0^{v_i} yg(y)\partial y \tag{2.26}$$

As the right hand side of the equality is independent of the mechanism $X$ the proof is complete. □

The Revenue Equivalence Principle shows the price only depends on the distribution of the buyer's valuations which means that the seller is paid the same regardless of the mechanism format differing only by a constant $K$ as a result of the integration [27].

### 2.2.7 *Impossibility Result*

Recall the four desirable properties discussed thus far.

- Individual Rationality (IR): No buyer (seller) would pay more (sell less) than its valuation (reservation) for the item it receives (sells).

$$v_i \geq p_b \; \forall \; b_i \in \mathbf{T}_b \tag{2.27}$$
$$r_j \leq p_s \; \forall \; s_j \in \mathbf{T}_s \tag{2.28}$$

- Incentive Compatibility (IC): Trader's dominant strategy is to simply report its private information truthfully

$$b_i = v_i \; \forall \; i \in \mathbf{N} \tag{2.29}$$
$$s_j = r_j \; \forall \; j \in \mathbf{M} \tag{2.30}$$

- Efficiency: Auction results in maximum gains from trade ensuring all efficient trades realized.

$$A(\mathbf{b}) = \mathbf{\Gamma} \tag{2.31}$$

- *Weak* Budget Balanced (BB): The Auction does not run a deficit.

$$p_b \geq p_s \tag{2.32}$$

An important result for mechanism design concerning these properties came from Myerson and Satterthwaite proving that for bilateral trade there is no mechanism that is IR, IC, BB, and efficient which is proved using the revenue equivalence principle [70].

**Impossibility Result Proposition:** *Under the assumption of bilateral trade, a mechanism cannot simultaneously be individually rational (IR), incentive compatible (IC), budget balanced (BB), and produce the ideally efficient allocation* [49]

*Proof.* Given a vector of buyer valuations and seller reservation prices $(\mathbf{v}, \mathbf{r})$, consider the VCG mechanism, $X_{VCG}$, from equation 2.19 which was proved to be IR, IC, and ideally efficient; but runs at a budget deficit. Consider some other mechanism, $X$ with different allocation and payments rules that is able to achieve the same economic properties of IR, IC, and ideally efficient. In IC mechanisms, traders are incentivized to truthfully reveal their private valuation and reservation prices through their bids resulting in the bidding strategy for both mechanisms being identical. By the revenue equivalence principle the expected payments for buyers, $p_b$, of $X$ and $X_{VCG}$ differ by exactly a constant $K$ due to the result of the integral on the right hand side of equation 2.26. Applying the revenue equivalence to opposite side of the market results in the prices received by seller, $p_s$, to differ by exactly a constant $L$.

Given IR, payments must be non-negative requiring $K \leq 0$ and $L \geq 0$ and therefore the expected difference between $X$ and $X_{VCG}$ is an additional deficit, $L - K \geq 0$, scaled by the number of efficient trades, $k$. Regardless of the allocation and payment rules for mechanism $X$, because it holds the properties of IR, IC, and is ideally efficient, the revenue can be written as a relation to that of the VCG mechanism according to equation 2.33.

$$\Pi(X) = \Pi(X_{VCG}) - k \times (L - K) \tag{2.33}$$

Given the VCG mechanism runs at a budget deficit equation 2.33 shows that mechanism $X$ also runs at a budget deficit. For bilateral trade, any other IR, IC, and ideally efficient mechanism $X$ also runs a budget deficit proving the proposition. □

This Impossibility result requires the mechanism designer determine which property(ies) they wish to sacrifice. Different lines of research exist exploring these trade-offs in various mechanisms under a variety of circumstances. The approach taken here involves mechanisms that

are IC as this is an extremely desirable property for a mechanism designer to reason about the auction's outcome and determine the truthful valuation and reservation prices of participates. According to Roughgarden, the tricky part in designing direct-revelation Dominant Strategy IC [Incentive Compatible] mechanisms is the careful coupling of the allocation and payment rules [82].

### 2.2.8 *Asymptotic Efficiency*

Following his contribution [70] to the famous impossibility result, Mark Satterthwaite together with Steven Williams, recognized that if ideal efficiency must be sacrificed, it might be regained in the limit as the markets grow sufficiently large [85].

The lower bound for valuation prices is denoted $\underline{v}$ while the upper bound for valuation prices is denoted $\overline{v}$. Assume all buyer's private valuations, $v_i$, are independently and identically distributed (i.i.d). Denote the cumulative distribution function $F_b(v)$ such that $F_b(\underline{v}) = 0$ and $F_b(\overline{v}) = 1$.

Likewise, the lower bound for reservation prices is denoted $\underline{r}$ while the upper bound for reservation prices is denoted $\overline{r}$. Assume all seller's private reservation prices, $r_j$, are independently and identically distributed (i.i.d). Denote the cumulative distribution function $F_s(r)$ such that $F_s(\underline{r}) = 0$ and $F_s(\overline{r}) = 1$.

There are six possible arrangements of the relationship between each of these bounds.

- $\underline{r} < \overline{r} < \underline{v} < \overline{v}$
- $\underline{r} < \underline{v} < \overline{r} < \overline{v}$
- $\underline{r} < \underline{v} < \overline{v} < \overline{r}$
- $\underline{v} < \underline{r} < \overline{v} < \overline{r}$
- $\underline{v} < \underline{r} < \overline{r} < \overline{v}$
- $\underline{v} < \overline{v} < \underline{r} < \overline{r}$

A potential efficient trade can only occur if the buyer and seller distributions overlap as represented by equation 2.34.

$$\max(\underline{r}, \underline{v}) < \min(\overline{v}, \overline{r}) \tag{2.34}$$

If the distributions do not overlap, there is no possibility for an efficient trade to occur. According to Definition 32, an outcome that does not produce any trades when no efficient trades are available is still considered ideally efficient, as all possible efficient trades are realized.

**Definition 41** : *Given overlapping buyer valuation and seller reservation price distribution functions according to equation 2.34, a mechanism is* ***asymptotically efficient*** *if the relative measure of efficiency e approaches 1 as the limit of the market size approaches infinity,* $\lim_{(\min(m+n))\to\infty} e = 1$. *Given there is no overlap such that equation 2.34 does not hold, a mechanism is* ***asymptotically efficient*** *if no trades occur and the relative measure of efficiency e approaches 0 as the limit of the market size approaches infinity,* $\lim_{(\min(m+n))\to\infty} e = 0$. [85].

It is not sufficient for one side to remain finite with the other side of the market approaches infinity. Asymptotic efficiency relies on additional realized efficient trades to mask any efficiency lost that prevents the mechanism from achieving ideal efficiency.

Re-scoping the aim to this concept of asymptotic efficiency; the goal becomes to have the efficiency lost account for a negligible amount of the overall efficiency as the market and number of traders grow. As the limit of the number of winning traders approach infinity, the achieved efficiency approaches the ideal efficiency, or the ratio of achieved efficiency to ideal efficiency approaches 1. The mechanism proposed that achieves this new standard of asymptotic efficiency is titled K-Double Auction (K-DA) [85].

The K-DA specifies a pricing rule that can be implemented in either a discriminatory or non-discriminatory way, both of which utilize a predetermined parameter $K \in [0, 1]$ known to all traders to calculate the price to be paid from buyer to seller. In the discriminatory K-DA each bilateral trading pair implements their own price according to the predetermined parameter $K$.

The buyer bid $b$ acts as the upper bound for the price while the seller bid $s$ serves as the lower bound. The value of $K$ dictates which side of the market is favored when determining the price to be paid via equation 2.35.

$$p = K * (b) + (1 - K) * (s) \tag{2.35}$$

A value of $K = 0.5$ represents an unbiased budget. A value of $K > 0.5$ represents a mechanism that favors the sellers side as the price tends towards the buyer's bid, the higher of the two. In the case where $K \in \{0, 1\}$ either the seller or buyer unilaterally determines the price to be paid. While this does not produce an equitable outcome, it does create an auction that is IC for the side of the market that does not influence price. The side of the market that dictates the price does not achieve the IC property as their bids determine the price.

A non-discriminatory pricing option uses the least favorable trade (LFT) as determined by the order statistics and equations 2.12 and 2.13, repeated below.

$$b_k \geq s_k \tag{2.12}$$
$$b_{k+1} < s_{k+1} \tag{2.13}$$

The non-discriminatory K-DA for multiple buyers and sellers can thus be expressed in equation 2.36

$$X_{KDA}(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow (\{b_1, ..., b_k, s_1, ...s_k\}, \{p_K, p_K\}) \tag{2.36}$$

where

$$p_K = K * (b_k) + (1 - K) * (s_k) \tag{2.37}$$

The resulting allocation is not equal to the ideally efficient allocation $\mathbf{\Gamma}$ because the bids do not truthfully reveal the private valuations. As the price $p_K$ is dependent upon the traders' bids, they have an incentive to misrepresent their true valuation and reservation prices. Buyers biding less than their valuation drive down the price, while sellers bidding larger than their reservation drive the price up. This strategic behavior results in the inevitable and unfortunate loss of trades that should take place, but do not. A trade that would result in a net positive gain from trade is subject to be removed from the efficient allocation. This occurs when the misrepresented bids cause the increased seller's bids to surpass the decreased buyer's bids despite the buyers true valuations exceeding the sellers true reservation prices.

The inefficiency loss due to the difference between bids and the true private values is quantified as $O(1/m)$ with an associated loss in GFT of $O(\frac{1}{m^2})$ [85]. The K-DA mechanism is IC for one-side of the market if and only if $K = 0$ or $K = 1$, otherwise the mechanism is not truthful. While not ideally efficient, the inefficiency lost becomes negligible reasonably quickly as the market grows. The mechanism does achieve Individual Rationality (IR) as well as Budget Balance (BB), and in fact due to the single price actually achieves the more strict SBB.

## 2.3 RESEARCH QUESTIONS

Section 2.2.7 introduced the Myerson and Satterthwaite impossibility result which states that there is no mechanism that is IR, IC, BB, and ideally efficient [70]. Presented in Chapter 3, McAfee designed a mechanism that can circumvent this impossibility result. The core concept revolves around the idea of a trade reduction. By purposefully sacrificing the least efficient trade the mechanism sacrifices ideal efficiency

in pursuit of asymptotic efficiency. This McAfee Trade Reduction mechanism is prior-free, deterministic, IR, IC, and BB [60].

Segal-Halevi et al. use a variation of this trade reduction technique to introduce a mechanism that achieves *strong* budget balance (SBB) at the cost of determinism. Following their work they pose an open research question: "is there a deterministic mechanism that is strongly-budget-balanced (in addition to being prior-free, truthful, individually-rational and approximately-optimal)?" [87].

It is assumed that by "truthful", the authors mean that the mechanism achieves the desired economic property of incentive compatibility (IC). While the definition of "**approximately-optimal**" is left open, it can reasonably be taken to mean asymptotic efficiency.

The first research question is motivated by this open question:

1. Is there a normalized, prior-free, and asymptotically efficient trade-reduction mechanism with a bid monotonic allocation rule that achieves both strong budget balance (SBB) and is deterministic?

Three additional assumptions are made to constrain the open research presented by Segal-Halevi et al. The first scopes consideration to trade reduction based mechanism, the topic of this research. The second assumes that any mechanism must also be normalized according to Definition 37. A normalized mechanism states that losing bidders do not incur a payment. This is a critical element of the IR property.

Finally, the third assumes that any mechanism must have a bid monotonic allocation rule according to Definition 38. Bid monotonicity states that bids must preserve a given order, such as being non-decreasing or non-increasing. This definition is critical to both IC and efficiency as it ensures that the most competitive bids as assigned to the efficient allocation.

Modern applications introduced in 1.1 often consist of an imbalance between buyers and sellers due to limited resources in emerging technologies with many interested users/consumers. These over-saturated markets may result in one side of the market with a small number of traders that is sensitive to any loss in efficiency. Additionally markets in emerging technology may be initially small before the technology is proven and attracts a larger number of traders. Auction mechanism must be able to achieve asymptotic efficiency while minimizing the loss in efficiency in smaller markets. This leads to the second research question:

2. Can a trade reduction mechanism be designed that holds the economic properties of IR, IC, and BB that is resilient to market saturation in terms of efficiency performance?

It can be critical for emerging markets to generate a positive revenue, not just over time, but in each and every market. This extends beyond the requirement for BB which requires the mechanism be non-negative. This leads to the third research question:

3. Can an IR, IC, and BB trade reduction mechanism be designed to guarantee a non-negative revenue while also achieving asymptotic efficiency?

Mechanism design is about making deliberate trade-off decisions. Because of the Myerson and Satterthwaite impossibility result at least one of the four desired economic properties (IR, IC, BB, and ideal efficiency) must be sacrificed. This research focus on trade reduction mechanism which sacrifice ideal efficiency in favor of asymptotic efficiency.

More specifically this research focus on secondary trade-off decisions. After an IR and IC mechanism is able to achieve both BB and asymptotic efficiency a trade-off can be made between optimizing two different measures of mechanism performance: efficiency and revenue. This research explores the decision space between efficiency and revenue performance.

# 3

# REVIEW OF TRADE REDUCTION MECHANISMS

The three research questions presented scope down consideration to mechanisms that achieve all three of the following desired economic properties: individual rationality (IR), incentive compatibility (IC), and budget balance (BB).

In the absence of external incentive sources prompting trader participation, the property of IR proves essential for mechanism design. Failure to achieve IR removes the guarantee that participants receive a non-negative payoff based on their bids. The possibility of an outcome resulting in a negative utility, simply by participating, may dissuade traders from participating.

To ensure that players receive a positive utility according to their private valuation and reservation prices requires the property of IC. By incentivizing traders to act in their own best interest and truthfully reveal their private information, markets can avoid potential market failure. The IC property is also extremely important to avoid market manipulation as well as eliciting the information required to clear markets with ideal or near ideal efficiency.

Mechanism may need incentivize the markets existence itself by guaranteeing the auction will not run a deficit (require funds be injected into the market) via the budget balance property (BB). This property is essential for mechanism design as failure to achieve BB undermines the sustainability of the auction.

Conflict occurs when it is desirable for the auction to hold all four properties: IC, IR, BB, and ideal efficiency due to the famous impossibility result for bilateral trade from [70]. The concern of the mechanism design field is the trade off between satisfying the properties that are valued most by the market in which the mechanism is to be implemented, and optimizing the desired properties that must be sacrificed.

As sacrificing IC often includes a sacrifice in efficiency, optimizing efficiency while maintaining the desired properties of IR, IC, and BB is a benchmark for performance.

Additionally, sacrificing ideal efficiency still leaves flexibility for efficiency to be maximized and allows auctions to clear with high rates of efficiency.

This chapter presents trade reduction mechanisms designed to truthfully illicit that private information in order to maximize efficiency subject to the economic properties of IR and BB.

## 3.1 MCAFEE TRADE REDUCTION

The McAfee trade reduction (TR) mechanism [60] provides an escape from the impossibility result [70]. This deterministic mechanism was designed to achieve IR, IC, and BB while sacrificing at most the least favorable trade (LFT), effectively reducing the ideally efficient trading allocation by no more than the least competitive trading pair. While not guaranteed, this mechanism has the potential to not reduce the ideally efficient trade at all.

The McAfee trade reduction mechanism was the first to achieve IR, IC, and BB while providing some efficiency guarantee. This trade reduction technique serves as the foundation for all other mechanisms that aim to achieve these economic properties, to include this line of research.

As at most the LFT is reduced, the TR mechanism provides a minimal guarantee equal to $1 - \frac{1}{k}$ fraction of the realized **optimal** GFT, where $k$ continues to be defined given the trader order statistics from equation 2.10 and equation 2.11 repeated here

$$b_1 \geq b_2 \geq ... \geq b_n \tag{2.10}$$

and

$$s_1 \leq s_1 \leq ... \leq s_m \tag{2.11}$$

and satisfying the efficient number of traders by equations 2.12 and 2.13 as defined by the allocation rule $k \leq \min\{m, n\}$ repeated here.

$$b_k \geq s_k \tag{2.12}$$

and

$$b_{k+1} < s_{k+1} \tag{2.13}$$

This pioneering mechanism is able to obtain asymptotic efficiency. As the number of traders increase, the value lost from the least favorable trade becomes inconsequential to the measure of market efficiency. However when the efficient number of traders $k = 1$ this mechanism provides no guarantee at all. The TR mechanism has become the foundation for the line of research concerned with designing IR, IC and BB mechanisms.

The mechanism introduces a fictitious buyer defined by equation 3.1 and seller defined by equation 3.2 for the case in which either all buyers, all sellers, or all traders are efficient traders.

$$b_{n+1} < b_n \tag{3.1}$$
$$s_{m+1} > s_m \tag{3.2}$$

Definition 32 characterizes the ideally efficient set of traders as buyers and sellers 1 through $k$. The fictitious buyer bid $b_{n+1}$ is denoted the lowest possible valuation while the fictitious seller bid $s_{m+1}$ denotes the highest possible reservation price.

### 3.1.1 *Mechanism Description*

Conceptually the trade reduction mechanism reduces $\mathbf{\Gamma}$ from length $k$ to some $A(\mathbf{b}, \mathbf{s}) = \mathbf{T}$ of length $l \leq k$. This requires a proper payment rule to enforce the desired mechanism properties of IR, IC, and BB. Define

$$p_0 = \frac{1}{2}(b_{k+1} + s_{k+1}) \tag{3.3}$$

Recall an allocation rule $A$ is defined by $A : (\mathbf{b}, \mathbf{s}) \rightarrow \mathbf{T} \dot{\cup} \overline{\mathbf{T}}$; a payment rule $P$ is defined by $P : (\mathbf{b}, \mathbf{s}) \rightarrow \mathbf{p} = (\mathbf{p_b}, \mathbf{p_s})$; and a mechanism defined by $X(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow [\mathbf{T} \dot{\cup} \overline{\mathbf{T}}, \mathbf{p}]$.

The McAfee mechanism $X_{Mc}$ is defined as follows.

$$X_{Mc}(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow \begin{cases} (\mathbf{\Gamma}, \{p_0, p_0\}) & p_0 \in [s_k, b_k] \\ (\mathbf{\Gamma} - \{b_k, s_k\}, \{b_k, s_k\}) & p_0 \notin [s_k, b_k] \end{cases} \tag{3.4}$$

The coupling of $A$ and $P$ in this way produces an efficient mechanism when $p_0 \in [s_k, b_k]$, and only looses out on a single trade when $p_0 \notin [s_k, b_k]$. Beyond that the trade that is lost is the least efficient trade as $b_k$ and $s_k$ are the least efficient traders. The case $p_0 \notin [s_{(k)}, b_{(k)}]$ requires a budget balancer to operate and absorbs a revenue of $p_b - p_s$.

### 3.1.2 *Property Proofs*

The McAfee mechanism, $X_{Mc}$, meets three of the four major desired economic properties.

**Lemma 1** : *The McAfee mechanism, $X_{Mc}$, satisfies the property of Individual Rationality* [60].

*Proof.* Consider the condition $p_0 \in [s_k, b_k]$. All efficient traders (buyers and sellers) trade at $p_0$ which by equation 3.3 lies strictly $b_{k+1} > p_0 > s_{k+1}$. Because of equations 2.10 and 2.11 all efficient buyers, $b_1, ..., b_k$ pay less than their bid and sellers $s_1, ..., s_k$ receive more than their bid:

$$b_1 > ... > b_k > p_0 > s_k > ... > s_1 \tag{3.5}$$

Equation 3.5 satisfies IR on both sides of the market.

Considering $p_0 \notin [s_k, b_k]$, traders $s_k$ and $b_k$ are removed from trading. Equation 2.10 states $b_{k-1}$ and all subsequent trading buyers are greater than the price to be paid, $b_1, ..., b_{k-1} \geq b_k = p_b$. Similarly, equation 2.11 states $s_{k-1}$ and all subsequent sellers are less than or equal to the price to be received, $s_1, ..., s_{k-1} \leq s_k = p_s$ hereby satisfying IR in both cases. □

Definitions 37 and 38 outline the components of a normalized mechanism with a bid monotonic allocation rule. These mechanism characteristics ensure winning bids do not leave the efficient allocation by increasing their bid and leads to the following crucial observation paving the way for Theorem 1.

**Observation 1** *Let $\boldsymbol{A}$ be a bid monotonic allocation rule, let $b_i \in \boldsymbol{b}$. If there exists a bid $\hat{b}_i$ for $i$ such $b_i \in \boldsymbol{T}_b$ and $\hat{b}_i$ for $b_i \notin \boldsymbol{T}_b$ then there exists a critical value $c_i$ such that $i$ wins if they bid $b_i > c_i$ and $i$ looses if they bid $b_i < c_i$; where $c_i$ is independent of $b_i$*

Notice this does not include the situation in which $b_i = c_i$ as the payoff would be exactly equal to zero as with all other losing traders given a normalized mechanism.

**Theorem 1** : *A normalized mechanism $X(A, P)$ is IC iff $A$ is bid monotonic and each trader $i, j \forall b_i, s_j \in \boldsymbol{T}_b, \boldsymbol{T}_s$ pays their critical value $p_b = c_i$ and $p_s = c_j$*

The following proof is adopted from Babaioff [7].

*Proof. Case if*

Assume that $X$ is normalized, $A$ is bid monotonic and buyer $i$ with valuation price $v_i$ pays $c_i$ if $b_i$ is allocated to the winning set, $b_i \in \mathbf{T}_b$. Proving IC involves showing that $i$ cannot improve their utility my misreporting their value. Consider the case in which buyer $i$ wins the auction by bidding his true valuation price $v_i$. If $i$ bids untruthfully and loses, then he gets zero utility, which by IR cannot be better than his utility with a truthful bid. If $i$ bids untruthfully and wins the auction, then since he still pays his critical value $c_i$ his utility remains the same.

Considering the case in which buyer $i$ loses the auction by bidding truthfully. His utility is zero and $v_i \leq c_i$ by Observation 1. If $i$ bids untruthfully and loses, his utility remains zero. If buyer $i$ bids untruthfully and wins, his utility is $v_i - c_i \leq 0$. In both cases, it is shown that a buyer cannot improve his utility by bidding untruthfully, thus proving that $X$ is IC for the buyer side [7].

The argument works analogously for the seller side given seller $j$ with reservation price $r_j$ who when paying their critical value $c_j$ result in an allocation to the winning set $\mathbf{T}_s$. When allocated to the losing set, seller $j$'s utility is zero. If seller $j$ bids untruthfully and wins, his utility is $c_j - r_j \leq 0$.

*Case only if*

Assume that the auction is normalized and IC (thus also IR) and its allocation rule is bid monotonic. To prove that each buyer $i$ with valuation price $v_i$ must pay his critical value $c_i$; assume that $p_b \neq c_i$ for some trader $i$. If $p_b > c_i$, then if $p_b > v_i > c_i$ and $i$ bids truthfully, then by Observation 1 $i$ wins but his utility is $v_i - p_b < 0$ which contradicts IR. If $p_b < c_i$, then if $p_b < v_i < c_i$ by Observation 1 $i$ loses and has zero utility if he bids truthfully. If $i$ misreports his value by bidding $b_i > c_i$, he would win the auction and has utility $v_i - p_b > 0$ which contradicts incentive compatibility.

The argument works analogously for seller $j$ by inverting the inequalities.

For normalized and IC mechanisms, the allocation rule which is bid monotonic uniquely defines the critical values for all the traders and thus the payments [7]. □

**Lemma 2** : *The McAfee mechanism, $X_{Mc}$, satisfies the property of Incentive Compatibility* [60].

*Proof.* All losing traders in the McAfee mechanism, $X_{Mc}$, do not pay or receive anything satisfying Definition 37 for a normalized mechanism. Given a trader is in the final allocation, a trader with a more competitive bid will also be in the final allocation. According to Definition 38 this allocation rule is bid monotonic.

Under the condition $p_0 \in [s_k, b_k]$, the critical value for buyers and sellers is the same $c_i = c_j = p_0$. All buyers above (sellers below) this critical value trade and pay their critical value. When $p_0 \notin [s_k, b_k]$ the critical value for the buyers side is $c_i = b_k$. The critical value for the sellers side $c_j = s_k$. All buyers above (sellers below) this critical value trade and pay their critical value. By theorem 1, $X_{Mc}$ is IC. □

As at most one trade is reduced from the efficient allocation, $X_{Mc}$, converges at a rate of $1 - \frac{1}{k}$ and is therefore asymptotically efficient [60].

**Lemma 3** : *The McAfee mechanism, $X_{Mc}$, satisfies the property of Budget Balance* [60].

*Proof.* A uniform price is implemented under the condition $p_0 \in [s_k, b_k]$ which satisfies not only BB but the more strict strong budget balance (SBB) also. When $p_0 \notin [s_k, b_k]$ the efficient traders $s_k$ and $b_k$ are removed from trading and used as prices for their markets, $b_k = p_b$ and $s_k = p_s$. Equation 2.12 ensures that for efficient traders the buyer bid exceeds or is equal to the seller bid thus ensuring non-negative revenue, $\Pi_{X_{Mc}} = p_b - p_s \geq 0$. □

**Definition 42** : *A **Prior-free** mechanism is able to implemented in a market without any assumptions on the distributions of either the buyer's valuation nor the seller's reservation prices* [86].

**Definition 43** : *A **Deterministic** mechanism acts as a function that takes in the actions of the traders (any vector of bid reports) and produces a single outcome that does not depend on the realization of a randomization device* [29].

**Theorem 2** : *The McAfee mechanism, $X_{Mc}$, is prior-free, deterministic, and satisfies the properties of IR, IC, asymptotically efficient, and BB.*

*Proof.* The McAfee mechanism, $X_{Mc}$, is deterministic by Definition 43 because it does not insert randomization into any part of the mechanism. The rules of the mechanism are not dependent on the prior distributions of the buyer's valuation or seller's reservation prices and is able to operate "prior-free". Lemma 1 proves the mechanism is IR, Lemma 3 proves the mechanism is BB, and Lemma 2 proves the mechanism is IC. At most one trade is reduced from the efficient allocation causing $X_{Mc}$ to converge at a rate $1 - \frac{1}{k}$ proving the mechanism is asymptotically efficient [60]. □

## 3.2 RANDOMIZED $\alpha$ REDUCTION

Babaioff and Nissan extended the McAfee mechanism, $X_{Mc}$, to achieve the more strict strong budget balance (SBB) property, which requires the revenue to be exactly zero rather than merely non-negative [7]. This leaves all gains inside the market rather than absorbed by a third party auctioneer. The authors were also able to reduce $X_{Mc}$ to a base trade reduction (TR) mechanism, $X_{TR}$ which does not incorporate the price, $p_0$ from equation 3.3 thereby removing the possibility of an ideally efficient allocation.

$$X_{TR}(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow (\mathbf{\Gamma} - \{b_k, s_k\}, \{b_k, s_k\}) \tag{3.6}$$

As a reduction of McAfee, The TR mechanism from equation 3.6 is deterministic, operates prior-free, and maintains the same desired economic properties of IR, BB, IC while achieving asymptotic efficiency. When $X_M c$ resulted in an ideally efficient allocation, a single price was used resulting in a revenue of exactly zero. The reduced trade reduction mechanism, $X_{TR}$, uses two prices and produces a budget surplus in each market with the rare exception of when $b_k = s_k$. This allows the authors to combine the budget surplus TR mechanism, $X_{TR}$, with the budget deficit VCG mechanism, $X_{VCG}$, from equation 2.19 to perfectly balance each other and produce a SBB mechanism [7].

### 3.2.1 *Mechanism Description*

The key concept in the non-deterministic (randomized) *$\alpha$Reduction* double auction mechanism, $X_\alpha$, is to use the knowledge of the distributions $F(v_i)$ and $G(r_j)$ along which the buyers' valuation prices and sellers' reservation prices fall. To balance the budget surplus of the TR mechanism with the budget deficit of the VCG mechanism the authors introduce a hyperparameter, $\alpha \in [0, 1]$. This hyperparameter becomes a function of the distributions, $\alpha = \alpha(F(v_i), G(r_j))$ and can be implemented in such a way that choosing the right value for $\alpha$ creates an expected revenue $\Pi = 0$ thereby strengthening the Budget Balance (BB) property from *weak* BB ($\geq 0$) to strong budget balance (SBB) in expectation (ex-ante).

$$X_\alpha(A, P) : (\mathbf{b}, \mathbf{s}) = \\ \alpha[X_{VCG}(A, P) : (\mathbf{b}, \mathbf{s})] + (1 - \alpha)[X_{TR}(A, P) : (\mathbf{b}, \mathbf{s})] \quad (3.7)$$

Due to the incorporation of randomization an additional definition is required for IC.

**Definition 44** : *A non-deterministic mechanism, X, is universally incentive compatibility if the traders are incentived to truthfully reveal their private valuation and reservation prices even if they know the randomization result.*

As both the TR and VCG mechanisms are IR and IC, $X_\alpha$, is also IR and universally IC regardless of the value of $\alpha$. Similar to McAfee as at most one trade is reduced from the efficient allocation $X_\alpha$ is asymptotically efficient. The *$\alpha$Reduction* **double auction mechanism** is not however deterministic and does require knowledge of the traders prior distributions. $X_\alpha$ does have the possibility of achieving SBB, if desired, depending on the selection of $\alpha$ [7].

Consider the following example implementing a trade reduction:

In the example above in Table 3.1 the efficient number of traders $k = 2$. The trade reduction reduces that trade size determined by the allocation rule to be $l = k - 1 = 1$. In this example all GFT are absorbed my the central auctioneer leaving the traders a payoff of

Revenue Example

| Seller bids | $p_s$ | $p_b$ | Buyer bids |
|---|---|---|---|
| 14 | 14 | 25 | 25 |
| ~~14~~ | - | - | ~~25~~ |
| 23 | - | - | 12 |

Table 3.1: Example: revenue

0. This shows that in many cases a significant portion of the GFT is absorbed by the market maker leaving little to none for the traders. This can be seen as leaving "money on the table, since the price paid by the buyers may be higher than the price paid to the sellers" [87]. If instead a single price could be used $p \in [s_k, b_k]$ then all GFT could be returned to the traders.

### 3.2.2 *Mechanism Performance*

The authors continue by comparing their new randomized mechanism with that of the TR mechanism, McAfee mechanism, and VCG mechanism by analyzing the revenue generated by each mechanism compared to the efficiency the mechanism achieves. The authors show four values of $\alpha$ for their new $\alpha$ Reduction mechanism: $\alpha \in \{0.25, 0.5, 0.75, \alpha^*\}$. The variable $\alpha^*$ represents the value of $\alpha$ required for an expected budget balance exactly equal to zero, $\Pi(X_\alpha = 0$. Buyer and seller private valuations and reservations prices are distributed uniform $[0, 1]$. The authors the behavior in two markets, the first with an even distribution of buyers and seller (25 buyers - 25 sellers) seen in Figure 3.1.

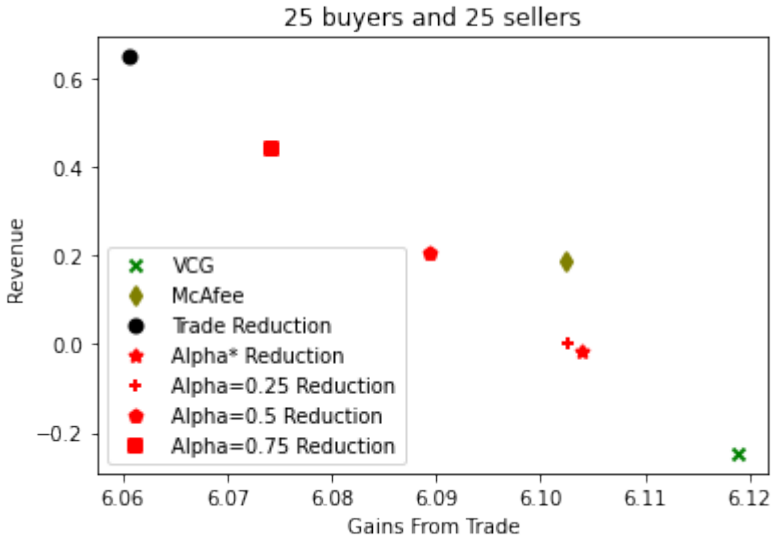



Figure 3.1: Mechanism Performance: 25 buyers-25 sellers (Reproduced)

The second market in Figure 3.2 inflates the buyers by increasing the amount to 50 and reduces the market to containing only 5 sellers.

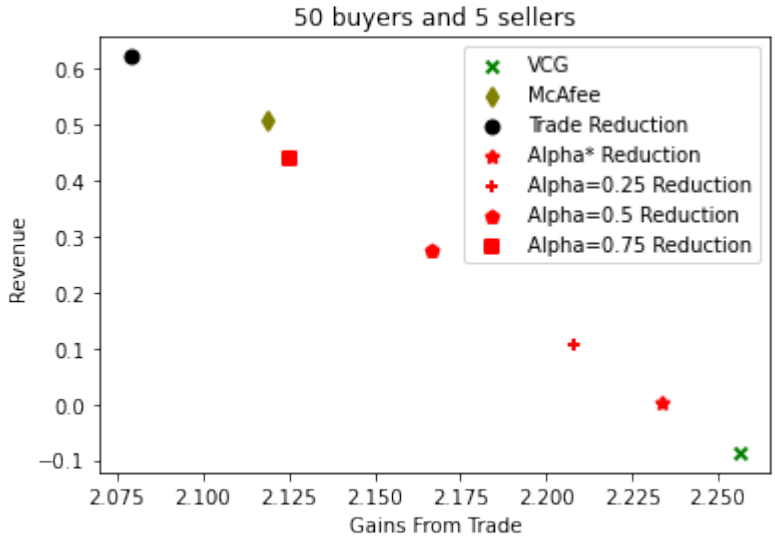



Figure 3.2: Mechanism Performance: 50 buyers-5 sellers (Reproduced)

The ideally efficient allocation as produced by the VCG mechanism marks the maximum efficiency to be attained on average across the 1000 runs. As the TR mechanism immediately reduces a trade and gives no chance at clearing a market with all efficient traders, this marks a minimum guaranteed efficiency to be achieved. Even a market of this size results in the TR mechanism achieving a very high efficiency of more than a 99%. The x-axis in these graphs represent the total GFT (gains from trade) on average across all runs. Given a trade size $l$ produced from a given mechanism the equation to calculate the total market GFT becomes:

$$\Omega = \sum_{1}^{l} v_i - \sum_{1}^{l} r_j \tag{3.8}$$

The limitation with both of these randomized mechanisms occur in the single use case, that is when the mechanism is only to be implemented once, or a small number of times, as opposed to repeated market implementation. The $\alpha$ Reduction DA losses relevancy as the mechanism implemented is either the VCG or TR mechanism. As the randomization mechanisms are BB in expectation the single use case looses the guarantee of this desired economic property of budget balance. Additionally, if there is only one efficient trader, $k = 1$, the mechanism will always result in relative efficiency of exactly zero, $e = 0$. The TR mechanism will always reduce the only efficient trade resulting in a revenue $\Pi = 0$ thereby not affording any opportunity to run the VCG mechanism at a deficit; in other words when $k = 1, \alpha = 0$.

## 3.3 SINGLE ITEM SPECIFICATION: MIDA MECHANISM

The Multi Item-type Double Auction (MIDA) Mechanism, $X_{MIDA}$, can be applied to the specific single item setting and is the first mechanism to achieve the more strict *strong* budget balance economic property in a prior-free market [86]. This MIDA mechanism overcomes two main limitations of the randomized $\alpha$ reduction mechanism, $X_\alpha$, proposed in Section 3.2 [7].

The first limitation of $X_\alpha$ is the mechanism achieves SBB in expectation only and provides no guarantee for any one specific market. Markets that desire to take advantage of the more strict SBB property are limited to those that are repeated a sufficiently large amounts of times; and are exclusively concerned with long-term performance in expectation, rather than at each implementation of the mechanism on any given market.

The second limitation is the setting in which the mechanism is to be implemented as $X_\alpha$ requires knowledge of prior distribution along with buyers' valuation and sellers' reservations prices fall. This information may in many cases be unavailable, as with emerging technologies, or difficult and expensive to acquire. The authors overcome these applicability limitations by introducing a mechanism using a common technique in the mechanism design field, **random sampling** [86]. The newly introduced mechanism, $X_{MIDA}$, is implemented according to a process and does not require knowledge of the prior distributions and achieves SBB in every market in which the mechanism is to implemented while maintaining the properties of IR, IC, and asymptotic efficiency.

### 3.3.1 *Random Sampling*

The use of randomization in sampling is a common technique in the mechanism design field and can be segregated into three distinct settings. The first and most restrictive is the *Bayesian* setting, in which the assumption is buyers' valuation and sellers' reservation prices are random variables drawn from a known probability distribution that is common knowledge to all traders as well as the mechanism designer. This setting is ideal for posted price mechanisms in which a trading price is determined independent of the trader's bids and is posted for all traders to determine whether they wish to participate in the auction or not. This is similar to how many modern supermarkets function. A price for the item is posted and the consumer selects which they wish to purchase at the given price according to their own private valuations. A posted price mechanism can be attractive because of it's simplicity allowing mechanisms to be implemented with a wide range of complex and unique trader utility function that may span across a large variety of markets. Posted price mechanisms have been shown to approximate the revenue of the optimal revenue-maximizing auction for single-parametric traders [16], [88], single unit-demand

traders [25], and multiple unit-demand traders [26], [98]. This setting allows the mechanism designer to approximate their desired outcome whether that be to maximize their own revenue, or to achieve a SBB mechanism in expectation as in the $\alpha$ reduction mechanism.

The second setting is *prior-independent* and is less restrictive than the *Bayesian* setting as the probability distributions are no longer assumed to be known. In this setting the assumption is buyers' valuation and sellers' reservation prices are random variables drawn from an unknown probability distribution. This setting allows for samples to be taken from the unknown distribution in order to estimate the true distribution. The main goal in this setting tends to focus on the number of samples required to achieve desired performance. An important step towards extending IR, IC, BB, and asymptotic efficient mechanism toward this setting [10]. In this mechanism the authors split the market into two halves and use each half to estimate the unknown probability distributions. Once a distribution is estimated it is then used to determine a trading price to be posted for both buyers and sellers. This concept of halving the market and using each half to determine a price for the other half enables mechanisms to designed that are incentive compatible as the traders' own bids are independent of the prices to be paid. The goal of this mechanism is to maximize revenue rather than strengthening the BB property to SBB to extend applicability of the mechanism.

Halving the market, first introduced by Baliga and Vohra [10], provided the foundation for the MIDA mechanism [86] allowing them to extend the applicability to the third *prior-free* setting. In this setting no assumptions are made regarding traders probability distributions and is the least restrictive and thereby the most challenging for mechanism design.

### 3.3.2 *Process of Mechanism Implementation*

The MIDA mechanism $X_{MIDA}$ is implemented according to a process and begins by injecting randomization by halving the bids into two distinct market: a right market $R_r$ and a left market $R_l$. Care is taken such that if the number of buyer bids $n$ (seller bids $m$) is even than each market receives exactly $n/2$ ($m/2$) bids to ensure the market size is exactly equal. If the number of bids are odd than one market will exactly one more bid than the other. This avoids random saturation of one side of the market in each half market. Each half market will then determine their optimal price window within their own market to be used in the other half. Denote the a buyer bid the right market $R_r$ as $b_i^r$ and a buyer bid in the left market as $b_i^l$. Similarly denote seller's bids as $s_j^r$ and $s_j^l$. The vector of bids in each half market becomes $\mathbf{s}^r \cup \mathbf{s}^l = \mathbf{s}$ and $\mathbf{b}^r \cup \mathbf{b}^l = \mathbf{b}$. As all buyers are assigned to one half or the other the number of buyers in each half sum to the total buyers in the whole market $n_r + n_l = n$. This is also true for the sellers $m_r + m_l = m$ [86].

Given the buyer, **b**, and seller, **s**, bids are organized by their order statistics defined in equations 2.10 and 2.11 the subsets for each half market are also organized by order statistics according to equations 3.9 through 3.12.

$$\mathbf{b}^r = \{b_1^r, ..., b_{n_r}^r\} \tag{3.9}$$
$$\mathbf{b}^l = \{b_1^l, ..., b_{n_l}^l\} \tag{3.10}$$
$$\mathbf{s}^r = \{s_1^r, ..., s_{m_r}^r\} \tag{3.11}$$
$$\mathbf{s}^l = \{s_1^l, ..., s_{m_l}^l\} \tag{3.12}$$

Should the highest buyer in the original whole market $b_1$ be assigned to the left half market this bid would be equal to $b_1 = b_1^l$. Should the second highest bid $b_2$ then be randomly assigned to the right half this bid would be equate to in notation $b_2 = b_1^r$ as it would be the most competitive bid in that half of the market.

The least favorable trade (LFT) in each half market are denoted $s_\kappa^r, b_\kappa^r$ for the right half and $s_\chi^l, b_\chi^l$ for the left half, such that

$$b_\kappa^r \geq s_\kappa^r \tag{3.13}$$
$$b_\chi^l \geq s_\chi^l \tag{3.14}$$

and

$$b_{\kappa+1}^r < s_{\kappa+1}^r \tag{3.15}$$
$$b_{\chi+1}^l < s_{\chi+1}^l \tag{3.16}$$

Prices are determined arbitrarily by each side of the market with the constraint that the price would lead to an ideally efficient allocation. Denote the price determined by the right side of the market $p_r$ and by the left side $p_l$. The bounds of the prices are determined by the least favorable bids in each side of the market: $p_r \in [s_\kappa^r, b_\kappa^r]$ and $p_l \in [s_\chi^l, b_\chi^l]$ [86].

There is a positive probability that a half market does not have any efficient trades at all. If there are no efficient buyers or sellers there is no restriction on the determination of the price. If there are efficient sellers but no efficient buyers willing to trade set the price equal to zero. If there are efficient buyers but no efficient sellers set the price to a sufficiently large value so as to drive the the demand to zero. Any price larger than the most competitive buyer prior to the split is sufficiently large [86].

Once determined prices are then posted to clear the other half market. Define a vector of eligible buyers in each half market $\beta^r = \{b_1^r, ..., b_\omega^r\}$ and $\beta^l = \{b_1^l, ..., b_\psi^l\}$ such that

$$b^r_\omega \geq p_l \tag{3.17}$$
$$b^l_\psi \geq p_r \tag{3.18}$$

and

$$b^r_\omega < p_l \tag{3.19}$$
$$b^l_\psi < p_r \tag{3.20}$$

Likewise define a vector of eligible sellers in each half market $\sigma^r = \{s^r_1, ..., s^r_\omega\}$ and $\sigma^l = \{s^l_1, ..., s^l_\psi\}$ such that

$$s^r_\omega \leq p_l \tag{3.21}$$
$$s^l_\psi \leq p_r \tag{3.22}$$

and

$$s^r_\omega > p_l \tag{3.23}$$
$$s^l_\psi > p_r \tag{3.24}$$

The process concludes with the final allocation determination. Traders are randomly reduced from the largest side of the market until the remaining number of eligible buyers match the number of eligible sellers. In other words if there are more eligible buyers then eligible sellers in a half market then the eligible buyers are randomly reduced by as many traders as required in order to be equal to the number of sellers in the half market. The randomization is required to maintain IC and discussed in the property proofs.

### 3.3.3 *Property Proofs*

The MIDA mechanism, $X_{MIDA}$, is prior-free by design [86].

**Lemma 4** : *The MIDA mechanism, $X_{MIDA}$ is ex-post individually rational (IR).*

*Proof.* Buyers are only eligible for trade if their bids exceed the posted price determined from the other half of the market as required by equations 3.17 and 3.18. Similarly eligible sellers leave the market if the posted price is not high enough as required by equations 3.21 and 3.22. □

**Lemma 5** : *The MIDA mechanism, $X_{MIDA}$ is ex-post Strong budget balanced (SBB).*

*Proof.* A single price is posted for buyers and sellers in each half of the market ensuring no revenue is required to be injected or absorbed. □

**Lemma 6** : *The MIDA mechanism, $X_{MIDA}$ is incentive compatible (IC).*

*Proof.* All losing traders in $X_{MIDA}$, do not pay or receive anything satisfying Definition 37 for a normalized mechanism. Given an eligible trader, a trader with a more competitive bid is also eligible for the final allocation. According to Definition 38 this makes the allocation rule bid monotonic.

The critical value for the left half of the market, $c^l$, is the price determined by the right side of the market, $p_r$, and is the same for both buyers and sellers, $c^l = c^l_i = c^l_j = p_r$. All buyers above (sellers below) this critical value are determined to be eligible traders who would pay their critical value if assigned to the final allocation.

Similarly, the critical value for the right half of the market, $c^r$, is the price determined by the left side of the market, $p_l$, and is the same for both buyers and sellers, $c^r = c^r_i = c^r_j = p_l$. All buyers above (sellers below) this critical value are determined to be eligible traders who would pay their critical value if assigned to the final allocation.

As the final allocation is determined by randomly reducing eligible traders if necessary, traders cannot influence the final allocation or price to be paid (received) by altering their bids. Traders thereby have no incentive to deviate from truth telling. As all buyers above (sellers below) their critical value that get to trade pay their critical value theorem 1 states $X_{MIDA}$ is IC. □

**Theorem 3** : *The MIDA mechanism, $X_{MIDA}$, is prior-free, and satisfies the properties of IR, SBB, and IC.*

*Proof.* The rules of the mechanism are not dependent on the prior distributions of the buyer's valuation or seller's reservation prices and is able to operate "prior-free". Lemma 4 proves the mechanism is IR, Lemma 5 proves the mechanism is SBB, and Lemma 6 proves the mechanism is IC. □

Further $X_{MIDA}$is asymptotically efficient and converges at a rate equal to $1 - O\sqrt{\frac{ln(k)}{k}}$ [86]. The authors identified three errors that may reduce the GFT. The first is sampling error, where one half market may indicate a significantly smaller number of efficient traders that exist in the other. The second is the pricing error, where the posted prices differ significantly from the **optimal** price leaving efficient traders outside of the final allocation as the posted price was set either too high or too low. This can be a result of the sampling error. The final

error is the competition error, where inefficient traders are included in the vector of eligible traders thereby increase the size of market and causing efficient traders to be randomly reduced [86].

While still asymptotically efficient, $X_{MIDA}$ converges at a much slower when compared to the deterministic McAfee mechanism $X_{MC}$ [60] which converges at $1 - \frac{1}{k}$ as at most one trade is reduced from the efficient allocation.

### 3.3.4 *MIDA Mechanism In Example*

Consider a balanced eight trader market consisting of four buyers and four sellers seen in Tables 3.2 and 3.3. This market contains up to $k = 3$ efficient trades as the three most competitive buyers are willing to pay more than the three most competitive sellers asking price.

MIDA Random Halving Example: Buyers

| Whole market notation | Buyer bids | Random halving assigned | Half market notation |
|---|---|---|---|
| $b_1$ | 25 | r | $b_1^r$ |
| $b_2$ | 23 | l | $b_1^l$ |
| $b_3$ | 14 | l | $b_2^l$ |
| $b_4$ | 12 | r | $b_2^r$ |

Table 3.2: Example: MIDA Random halving-buyers

MIDA Random Halving Example: Sellers

| Whole market notation | Seller bids | Random halving assigned | Half market notation |
|---|---|---|---|
| $s_1$ | 7 | l | $s_1^l$ |
| $s_2$ | 10 | r | $s_1^r$ |
| $s_3$ | 12 | l | $s_2^l$ |
| $s_4$ | 14 | r | $s_2^r$ |

Table 3.3: Example: MIDA Random halving-sellers

The MIDA mechanism, $X_{MIDA}$, first randomly splits the market in half. The results of the random split are also seen in the last column of Tables 3.2 and 3.3. This split results in a right half of the market seen in Table 3.4 and a left half of the market seen in Table 3.5.

Right half of the market

| Seller bids | Buyer bids |
|---|---|
| 10 | 25 |
| 14 | 14 |

Table 3.4: Right half of the market

Left half of market

| Seller bids | Buyer bids |
|---|---|
| 7 | 23 |
| 12 | 12 |

Table 3.5: Left half of market

The price produced by the right side of the market is $p_r = 14$ and the price produce by the left side is $p_l = 12$. The remaining eligible traders are seen in Tables 3.6 and 3.7. Ineligible traders are struck out.

Right half of the market

| Seller bids | $p_l$ | Buyer bids |
|---|---|---|
| 10 | 12 | 25 |
| ~~14~~ | 12 | 14 |

Table 3.6: Right half Ineligible traders

Left half of market

| Seller bids | $p_r$ | Buyer bids |
|---|---|---|
| 7 | 14 | 23 |
| 12 | 14 | ~~12~~ |

Table 3.7: Left half Ineligible traders

This leaves the right half of the market with one too many buyers and the left half of the market with one too many sellers. One eligible buyer must be randomly reduced from the right side of the market and one eligible sellers must be randomly reduced from the left side of the market. Remaining traders that are assigned to the winning allocation are seen in Tables 3.8 and 3.8 with losing traders struck out.

Right half of the market

| Seller bids | $p_l$ | Buyer bids |
|---|---|---|
| 10 | 12 | ~~25~~ |
| ~~14~~ | 12 | 14 |

Table 3.8: Right half winners

Left half of market

| Seller bids | $p_r$ | Buyer bids |
|---|---|---|
| 7 | 14 | 23 |
| ~~12~~ | 14 | ~~12~~ |

Table 3.9: Left half winners

The final allocation in this example results in two trades with a loss in GFT for the whole market $= 25 - 12 = 13$. The whole market GFT from the example is $23 + 14 - 7 - 10 = 20$. This results in an efficiency rate of $\frac{20}{33} = 0.61$. An alternate random halving could have resulted in the following two half markets seen in Tables 3.10 and 3.11 with prices determined by the opposite half.

Right half of the market

| Seller bids | $p_l$ | Buyer bids |
|---|---|---|
| ~~12~~ | 11 | 25 |
| ~~14~~ | 11 | 23 |

Table 3.10: Right half prices

Left half of market

| Seller bids | $p_r$ | Buyer bids |
|---|---|---|
| 7 | 20 | ~~14~~ |
| 10 | 20 | ~~12~~ |

Table 3.11: Left half prices

As seen by this alternative split of the same market in Tables 3.10 and 3.11, the possibility exists that no trades occur in either half. This particularly vulnerable with smaller markets. As shown in in [86] as the market grows each half of the market will better represent the other and the mechanism is able to achieve asymptotic efficiency.

## 3.4 STRONGLY-BUDGET-BALANCED-DOUBLE AUCTION

A pair of mechanisms entitled "Strongly Budget Balanced Double Auction" (SBBA) Mechanisms [87] improve on the efficiency of the MIDA mechanism [86] while maintaining the economic properties of IR, IC, and SBB and does not require knowledge on the prior distribution and is thus *prior-free*. The innovative design of the SBBA mechanism comes by combining the payment rule from VCG, $X_{VCG}$, shown in equation 2.19 with the allocation rule from the trade reduction mechanism, $X_{TR}$, shown in equation 3.6. The major contribution comes with the improvement in the expected asymptotic efficiency conversion rate equal to that of $X_{MC}$ as at most one set of traders (one buyer and one seller) are reduced from the trading allocation.

In the SBBA mechanism, $X_{SBBA}$, randomization is used similar to MIDA but is restricted to one side of the market only in the event a trade reduction is necessary. Unlike MIDA, at most one trade must be reduced leading to the expected efficiency of $1 - \frac{1}{k}$ similar to McAfee. According to the authors the shortcoming of $X_{SBBA}$ is the mechanism only holds in expectation due to the nature of the random reduction while $X_{Mc}$ holds in the worst case [87]. Additionally, as the LFT traders $s_k$ and $b_k$ are not guaranteed to be reduced due to the randomization of the reduction the efficiency from SBBA in the case of a trade reduction will always be less than or equal to that of McAfee.

### 3.4.1 *Mechanism Description*

Recall the VCG mechanism from equation 2.19

$$X_{VCG}(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow (\mathbf{\Gamma}, \{\max(s_k, b_{k+1}), \min(s_{k+1}, b_k)\}) \quad (2.19)$$

The key insight for creating the SBBA mechanism is too split the VCG payments and create two separate mechanisms. The first mechanism, $X_{SBBA1}$, uses a single price for both buyers and sellers, $p_1$ and is equal to the seller's price from the VCG mechanism. The second mechanism, $X_{SBBA2}$, which is mentioned but not explicitly introduced by the authors, also uses a single uniform price for both buyers and sellers, $p_2$ and is equal to the buyer's price from the VCG mechanism [87].

$$p_1 = \min(s_{k+1}, b_k) \quad (3.25)$$
$$p_2 = \max(s_k, b_{k+1}) \quad (3.26)$$

The VCG mechanism, $X_{VCG}$, operate on a budget deficit requiring the market to be subsidized in order to function. This is because the price the buyers pay is not sufficient to cover the price the sellers receive, $p_b = p_2 \leq p_s = p_1$. Splitting these prices up to create two mechanisms that are then combined with the trade reduction technique creates mechanisms that favor one side of the market. The first mechanism that implements the higher price according to equation 3.25 is more appealing to the sellers when compared to the second mechanism that implements the lower price according to equation 3.26 that is more appealing to the buyers.

The first mechanism, $X_{SBBA1}$, considers the relationship between $s_{k+1}$ and $b_k$ to dictate the price. The following two cases can occur.

- $s_{k+1} \leq b_k$
- $s_{k+1} > b_k$

In the first case $p_1 = \min(s_{k+1}, b_k) = s_{k+1}$ and acts as the critical value for both sides of the market. As sellers are organized according to their competitiveness as in equation 2.11 all efficient sellers are by definition $\{s_1, ...., s_k\} \leq s_{k+1}$. As the single-uniform price is acceptable for all efficient traders, $p_1 = s_{k+1} \in [s_k, b_k]$, the market clears $k$ traders as no trade reduction is necessary given. The second case implements an alternate price, $p_2 = \min(s_{k+1}, b_k) = b_k$, which requires buyer $b_k$ be reduced from trade so as to use their bid as the critical value for both sides of the market. Under this case the market clears $k-1$ traders leaving efficient buyers $\{b_1, ..., b_{k-1}\}$ assigned to the winning allocation. Sellers are organized according to their competitiveness as in 2.11; as buyer $b_k$ and seller $s_k$ are efficient traders, equation 2.12 requires $b_k \geq s_k \geq ... \geq s_1$. To maintain material balance a trade reduction is still required from the vector of efficient sellers $\{s_1, ..., s_k\}$. To maintain truthfulness a seller is selected at random and is just as likely to be the most competitive seller $s_1$ as it is $s_k$. Denote the random seller reduced from the winning allocation as $s_R \leq s_k$ the first SBBA mechanism can be written as

$$X_{SBBA1}(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow \begin{cases} (\mathbf{\Gamma}, \{s_{k+1}, s_{k+1}\}) & s_{k+1} \leq b_k \\ (\mathbf{\Gamma} - \{b_k, s_R\}, \{b_k, b_k\}) & s_{k+1} > b_k \end{cases} \tag{3.27}$$

The price $p_1$ implemented here is higher than $p_2$ and thus more appealing to sellers, though the possibility of being randomly reduced can be unappealing to highly competitive sellers. Whether the higher price covers up the risk of being reduced is dependent upon both the seller and the market. The key factors are the seller's private reservation prices, the prior distribution for buyers' valuation and sellers' reservation prices, and the size of the market (both in number of traders and ratio of buyers to sellers).

The second mechanism, $X_{SBBA2}$, operates analogously to the first but considers the relationship between $b_{k+1}$ and $s_k$.

$$X_{SBBA2}(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow \begin{cases} (\mathbf{\Gamma}, \{b_{k+1}, b_{k+1}\}) & b_{k+1} \geq s_k \\ (\mathbf{\Gamma} - \{b_R, s_k\}, \{s_k, s_k\}) & b_{k+1} < s_k \end{cases} \tag{3.28}$$

### 3.4.2 *Property Proofs*

The SBBA mechanisms, $X_{SBBA1}, X_{SBBA2}$, meet the same three of the four major desired economic properties as the previous mechanisms.

**Lemma 7** : *The first SBBA mechanism, $X_{SBBA1}$, satisfies the property of Individual Rationality* [87].

*Proof.* Under the condition all efficient traders (buyers and sellers) trade, the price, $p_1 = s_{k+1}$, = satisfying IR on both sides of the market $s_k < p_1 < b_k$.

When the price results in a trade reduction, $p_1 = b_k$, buyer $b_k$ and a random seller $s_R$ are removed from trading. Equation 2.10, defining buyer order statistics, states $b_{k-1}$ and all subsequent trading buyers are greater than the price to be paid, $b_1, ..., b_{k-1} \geq b_k = p_b$. Similarly, equation 2.11, defining seller order statistics, states $s_{k-1}$ and all subsequent sellers are less than or equal to the price to be received, $s_1, ..., s_{k-1} \leq s_k = p_s$ hereby satisfying IR in both cases. □

The argument works analogously for the second mechanism, $X_{SBBA2}$.

**Lemma 8** : *The first SBBA mechanism, $X_{SBBA1}$, satisfies the property of Incentive Compatibility* [87].

*Proof.* All losing traders do not pay or receive anything satisfying Definition 37 for a normalized mechanism. Given a buyer is in the final allocation, a buyer with a more competitive bid will also be in the final allocation. According to Definition 38 this allocation rule is bid monotonic for the buyer's side.

A seller, $j$, is assigned to the losing allocation when their bid is too larger to be competitive, $s_j \geq s_{k+1}$. Assigning a competitive seller's bid, $s_j \leq s_k$ to the winning allocation does not depend on their bid, $s_j$, but on the relationship between $s_{k+1}$ and $b_k$.

- If $s_{k+1} \leq b_k$ the probability that $s_j \leq s_k$ wins is 1.
- $s_{k+1} > b_k$ the probability that $s_j \leq s_k$ wins is $1 - \frac{1}{k}$ due to the randomized reduction.

In each of these cases, reducing their bid only decreasing their index as defined by the order statistics in equation 2.11. The winning probability remains unchanged resulting in a bid monotonic allocation rule for the seller's side [87].

A single uniform price is always implemented in each mechanism making the critical value for buyers and sellers is the same $c_i = c_j = p_1$. Under the condition $p_1 = s_{k+1}$ all buyers above (sellers below) this critical value trade and pay their critical value.

When $p_1 = b_k$ all buyers above this critical value trade and pay their critical value while a random seller is reduced. As in $X_{MIDA}$ the use of randomization makes it so that traders cannot influence the final

allocation or price to be paid by altering their bids. Traders thereby have no incentive to deviate from truth telling. All remaining sellers below this critical value trade and pay their critical value. By theorem 1, $X_{SBBA1}$ is IC. □

The argument works analogously for the second mechanism, $X_{SBBA2}$.

**Lemma 9** : *The SBBA mechanisms, $X_{SBBA1}, X_{SBBA2}$, satisfy the property of Budget Balance* [87].

*Proof.* A uniform price is implemented in both mechanisms satisfying not only BB but the more strict strong budget balance (SBB) also. □

**Theorem 4** : *The SBBA mechanisms, $X_{Mc}$, is prior-free and satisfies the properties of IR, BB, and IC while achieving asymptotic efficiency.*

*Proof.* By Definition 42 the SBBA mechanisms, $X_{SBBA1}, X_{SBBA2}$, are designed to be prior-free as the rules of the mechanism are not dependent on the prior distributions of the buyer's valuation or seller's reservation prices. Lemma 7 proves the mechanism is IR, Lemma 9 proves the mechanism is BB, and Lemma 8 proves the mechanism is IC. At most one trade is reduced from the efficient allocation causing $X_{SSBA1}$ and $X_{SBBA2}$ to converge at a rate $1 - \frac{1}{k}$ proving the mechanism is asymptotically efficient [87]. □

### 3.4.3 *Difficulties of Saturated Markets*

Not knowing the prior distribution is just one source of uncertainty in mechanism design, but it's not the only one. The frequency with which one side of the market dominates, whether buyers or sellers introduces another layer of unpredictability. This imbalance can lead to either buyer or seller saturation, where the market is skewed towards one side, further complicating the design and implementation of effective mechanisms.

Both the lack of prior distribution knowledge and the potential for market saturation must be addressed to achieve optimal outcomes. The challenge lies in designing mechanisms that perform well universally, whether in balanced markets with equal buyers and sellers or in markets saturated toward one side.

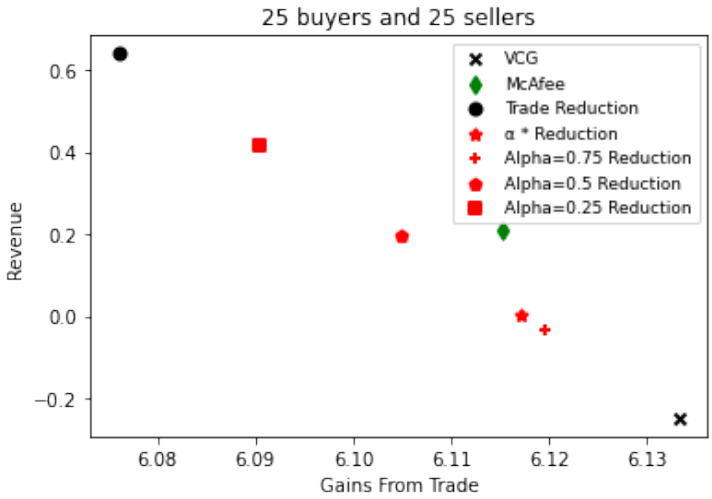



Figure 3.3: Efficiency Vs. Revenue (reproduced graph)

Figure 3.3 is a reproduction using the visualization techniques presented by Babaioff and Nissan and shows the comparison between revenue on the y-axis and efficiency on the x-axis. The figure was originally created to show efficiency performance of various mechanisms in an even, non-saturated, market with 25 buyers and sellers [7]. The difference in the version recreated in Figure 3.3 is markets were simulated 10,000 times as opposed to the 1,000 times originally [7].

The VCG mechanism represents the upper bound in efficiency due to it's economic property of ideal efficiency. The "Trade Reduction" mechanism from equation 3.6 represents the other extreme. This mechanism is used to increase revenue and avoid the use of a single price. A market that clears with a single price does not produce any revenue, positive or negative. In the trade reduction mechanism the least favorable trade is always reduced to allow for multiple prices, one for each side of the market.

The alpha reduction mechanisms introduces the hyperparameter $\alpha$ to combine VCG and TR mechanism. When alpha is equal to one the alpha mechanism reduces to the trade reduction mechanism. When alpha is set to zero the mechanism reduces to the VCG mechanism. A straight line can be drawn between the two mechanisms, such that the selection of alpha determines the specific point along that line.

The McAfee mechanism falls above this fictitious line, meaning the mechanism is either more efficient, or produces a higher revenue.

Figure 3.5 overlays the MIDA and SBBA mechanism that were introduced after the publication of the original figure. These mechanisms are designed to be strong budget balance, have a revenue of zero.

The strengthening of the BB economic property to SBB comes at a cost to efficiency. In an even market with equal numbers of buyers and sellers, both the SBBA mechanisms perform identically. They are also able to outperform the MIDA mechanism in terms of efficiency.

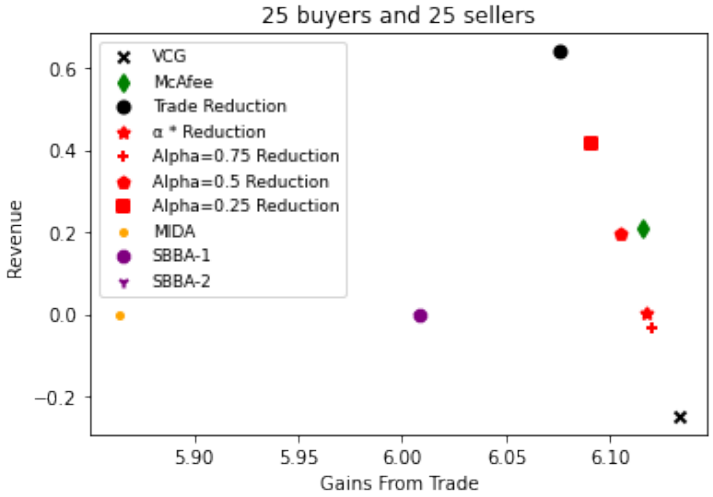



Figure 3.4: Overlay with SBB mechanisms

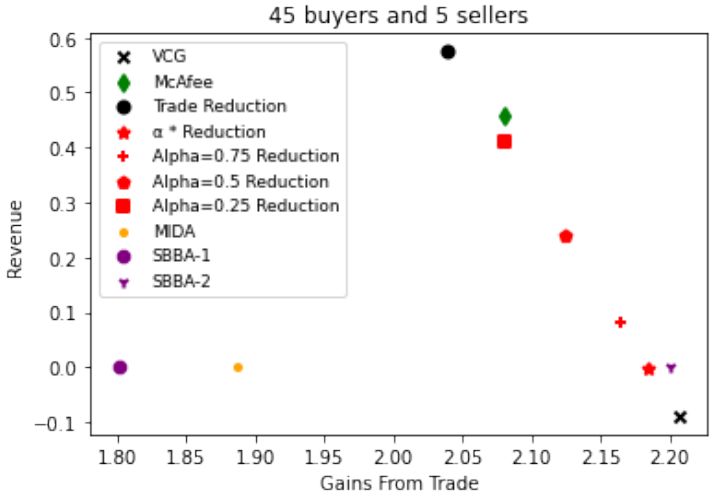
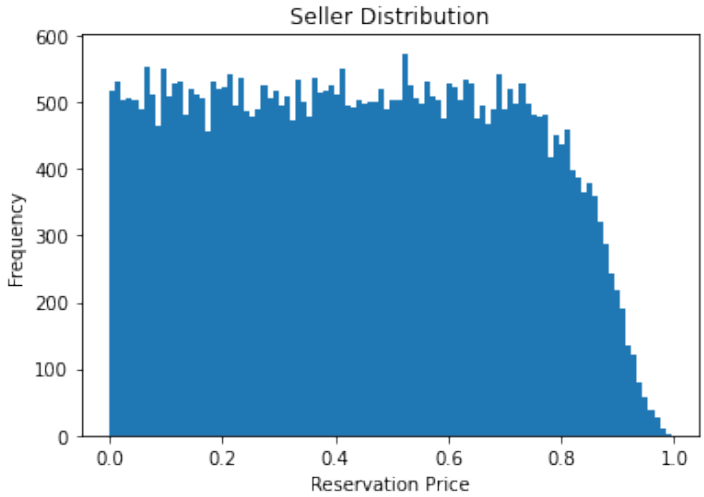



Figure 3.5: 45 buyers and 5 sellers

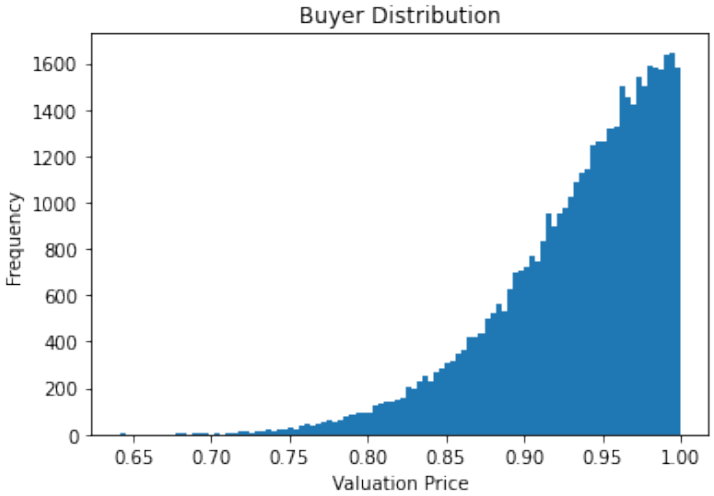
Performance changes as soon as the market becomes saturated. Using the same size in terms of total traders, Figure 3.5 shows performance in a buyer saturated market. Simulations were run using a normalized uniform distribution for both buyers and sellers. Market saturation cause a dichotomy in efficiency performance between the two SBBA mechanisms.

The distribution of efficient traders reflect the underlying distribution of private valuation and reservation prices. Overlapping the distribution of efficient buyers and sellers would result in a combined distribution that resembles a normalized uniform distribution. In a non-saturated market the drop off of sellers and subsequent increase in number of buyers should occur at the mid point (0.5) of the normalized space. In a saturated market this overlap is pushed to the side of the market that is saturated.

Figure 3.6 shows the distributions of efficient sellers and buyers from the same buyer saturated market in Figure 3.5. Increased demand raises the efficient clearing price due to a limited supply. The large number of buyers in the market relative to sellers pushes the most competitive buyers toward the top of the distribution as seen in Figure 3.6b. The sellers side of the market is largely unaffected by this phenomena as seen by the largely uniform nature of the distribution scene in Figure 3.6a.




(a) Distribution of efficient sellers with market size of 50




(b) Distribution of efficient buyers with market size of 50

Figure 3.6: Efficient Trader Distribution: 90% buyer

The dichotomy in efficiency performance between the two SBAA mechanism can be explained by two contributing factors. The first is the choice of price. The SBBA-2 mechanism implements a lower price, $p_2 = \max(b_{k+1}, s_k)$, compared to the SBBA-1 mechanism, $p_1 = \min(s_{k+1}, b_k)$.

Buyer saturation decreases variance among efficient buyers and increases it among efficient sellers. This narrows the gap between the least efficient buyer ($b_k$) and the most competitive inefficient buyer ($b_{k+1}$), while widening the gap between the least efficient seller ($s_k$) and the most competitive inefficient seller ($s_{k+1}$). As a result, the likelihood that $s_{k+1}$ falls between $b_k$ and $b_{k+1}$ decreases, reducing the chances of an ideally efficient allocation for the SBBA-1 mechanism (The SBBA-1 mechanism results in an ideally efficient allocation when $b_k \geq s_{k+1}$). Conversely, the likelihood that $b_{k+1}$ falls between $s_k$ and $s_{k+1}$ increases, improving the chances of an ideally efficient allocation for the SBBA-2 mechanism.

The second factor is randomized reduction of one side of the market. The SBBA-1 mechanisms randomly reduces a seller when a trade reduction is performed, while the SBBA-2 mechanism reduces a random buyer. As the efficient buyer's distribution has less variance, the least favorable buyer is likely to have a valuation similar to that of the randomly reduced buyer. The potential efficiency lost is constrained by the buyer's distribution. The increased variance in the efficient seller's distribution results in a larger potential for lost efficiency between the least efficient seller and the randomly reduced seller. In a buyer saturated market, the potential loss of efficiency by randomly reducing a seller is therefore larger compared to randomly reducing a buyer.

For the SBBA-2 mechanism, both the lower price and random reduction of the buyer side of the market results in a more efficient mechanism. Similar arguments apply to SBBA-1 in a seller saturated market.

The choice of mechanism depends on the market in which it will be implemented and the objectives it aims to achieve. The mechanism designer must make a trade off between the efficiency and revenue a mechanism produces.

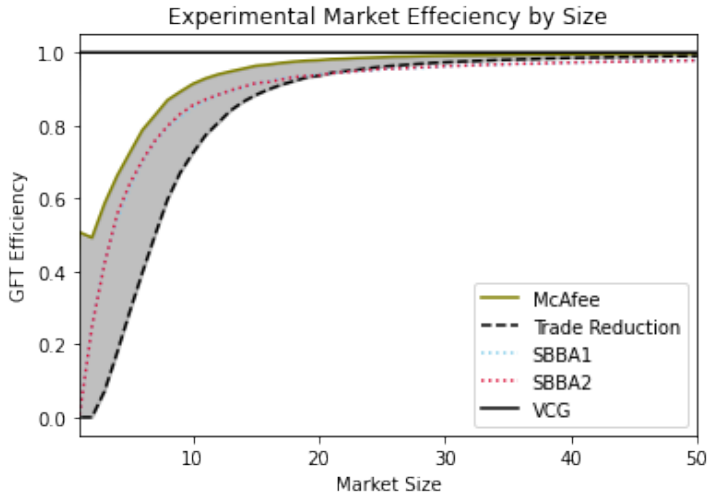



Figure 3.7: Efficiency (50% buyers)

Figure 3.7 shows the efficiency of all budget balanced mechanisms according the size of the market. Traders that enter the market are equally as likely to be assigned as buyers and sellers. Valuation and reservation prices are normalized and uniformly distributed. The shaded region between the McAfee mechanism and the trade reduction shows the trade-off space in terms of efficiency when budget balance is a desired economic property.

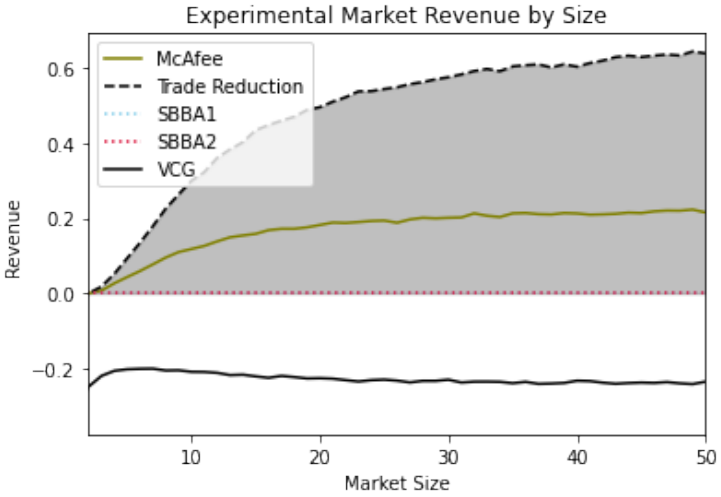



Figure 3.8: Revenue (50% buyers)

This can be compared against the shaded region marking the trade off in revenue in Figure 3.8.

Figures 3.9 and 3.10 show the efficiency and revenue performance respectively of buyer saturated markets. These markets are dominated by buyers where a trader entering the market has a 90% chance of being assigned as a buyer. The mechanisms bounding the trade off space change in saturated markets.

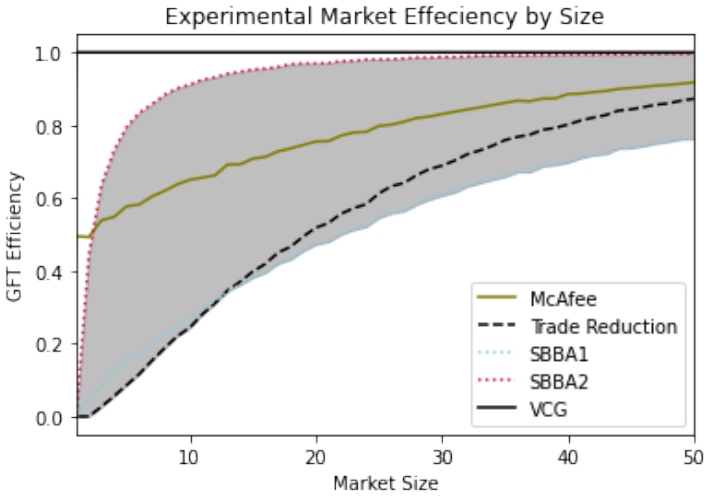



Figure 3.9: Efficiency (90% buyers)

Figure 3.9 shows the efficiency trade-off. The SBBA-1 mechanism sets the lower bound while the SBBA-2 sets the upper bound in a buyer saturated market. The revenue space, shown in Figure 3.10, continues to show the trade reduction mechanism as having the largest revenue.

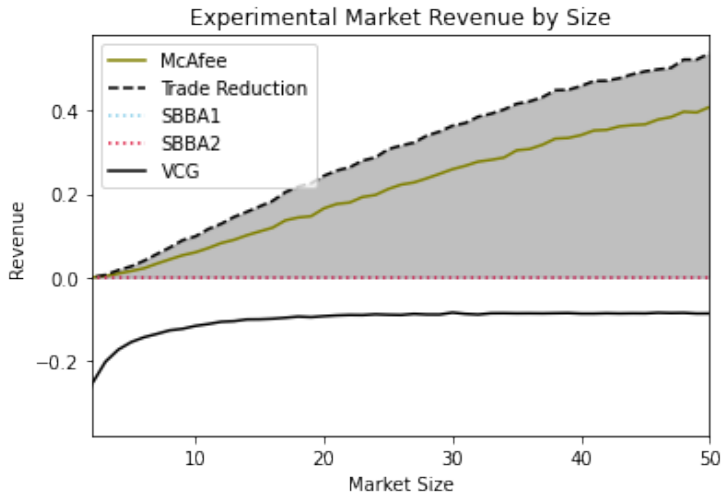



Figure 3.10: Revenue (90% buyers)

Under-saturated traders are likely to be fully absorbed in the efficient allocation. For example, if buyers significantly outnumber sellers, it is more likely that the entire supply from sellers will be purchased. The more buyers there are relative to sellers, the more likely it is that even the least competitive seller (the one with the highest reservation price) will find a buyer with whom to trade efficiently.

The McAfee mechanism aims to mitigate a lack of inefficient traders by introducing fictitious order statistics for $b_{k+1}$ and $s_{k+1}$. The SBBA mechanisms, however, cannot clear the market with ideal efficiency when inefficient traders do not exist. The SBBA pricing rules will immediately reduce the trade due to a lack of an inefficient buyer for $p_2 = \max(b_{k+1}, s_k)$ or lack of inefficient seller with $p_1 = \min(s_{k+1}, b_k)$.

This challenge is exacerbated in saturated markets. Continuing with 90 % buyer saturated market, simulation results in Table 3.12 illustrate when an immediate trade reduction is necessary due to the absence of an inefficient buyer to set the price.

In a buyer saturated market using $p_2 = \max(b_{k+1}, s_k)$ Table 3.12 shows that a market size of three will result in an immediate trade reduction 197 out of $10,00$ runs due to the lack of existence of an inefficient buyer $b_{k+1}$.

Buyer saturated markets present a significant challenge to the SBBA-1 mechanism with price $p_1 = \min(s_{k+1}, b_k)$. Due to the scarcity of sellers, it is much more likely that all sellers are part of the ideally efficient allocation. Table 3.13 shows the number of simulations (out of 10,000) that resulted in a trade reduction due the lack of an inefficient seller $s_{k+1}$. The first row shows markets sizes $1-10$ while the last row shows market sizes $41-50$.

| 50 Trader Market (10,000 runs) 90% buyer saturated | |
|---|---|
| Market size | Number of runs with all efficient buyers |
| 3 | 197 |
| 4 | 28 |
| 5 | 17 |
| 6 | 1 |
| 7 | 1 |
| 8+ | 0 |

Table 3.12: SBBA-2 reduction from lack of buyers

| Number of runs with all efficient sellers [50 Trader - 90% buyer saturated Market (10,000 runs)] | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | ones | | | | | | | | | |
| tens | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 0 | - | - | 8924 | 8797 | 8733 | 8611 | 8578 | 8514 | 8427 | 8424 |
| 1 | 8270 | 8174 | 8127 | 7993 | 7920 | 7945 | 7846 | 7778 | 7711 | 7580 |
| 2 | 7561 | 7541 | 7394 | 7326 | 7299 | 7191 | 7156 | 7045 | 6959 | 6935 |
| 3 | 6896 | 6781 | 6736 | 6728 | 6579 | 6544 | 6500 | 6444 | 6368 | 6324 |
| 4 | 6266 | 6176 | 6155 | 6186 | 5910 | 5997 | 5953 | 5929 | 5782 | 5707 |

Table 3.13: SBBA-1 reduction from lack of sellers

Table 3.13 illustrates that all available sellers are often efficient in a buyer-saturated market. With a market size of 50, over 55% of simulations show all sellers as efficient, leading to an immediate trade reduction as no inefficient seller ($s_{k+1}$) exists. A market size 20 shows that 75 % of runs result in all sellers acting as efficient sellers. This efficiency loss is compounded when an efficient buyer is randomly removed. As a result, the SBBA-1 mechanism's efficiency falls below that of the MIDA mechanism, as shown in Figure 3.5.

When applying auction mechanisms to markets with emerging technologies, it may be unclear whether the market will attract more buyers or sellers. In risk-averse markets, an argument can be made for using the MIDA mechanism, despite its lower asymptotic efficiency, depending on the size of the market it will be implemented in.

Each of two SBBA mechanisms are either extremely efficient or inefficient depending on which side of the market is over-saturated.

## 3.5 SUMMARY

This chapter has provided an in-depth literature review on specific trade reduction mechanisms, establishing a foundation for the research questions that will be addressed in the following chapters. Particular emphasis was placed on achieving the key economic properties of individual rationality (IR), incentive compatibility (IC), and budget balance (BB) while achieving asymptotic efficiency.

Trade reduction mechanisms, first introduced by McAfee [60], are foundational in mechanism design due to their ability to circumvent the Myerson-Satterthwaite impossibility result. By sacrificing only the least efficient trade, these mechanisms ensures asymptotic efficiency as market size increases. Table 3.14 succinctly highlights each of the specific trade reduction mechanisms introduced along with their ability to achieve certain economic properties. Mechanisms are listed in chronological order of publication.

Existing Mechanisms

| Mechanism | Budget Balanced | Asymptotic Efficiency | Prior free | Deterministic |
|---|---|---|---|---|
| McAfee | BB | $1 - \frac{1}{k}$ | Yes | Yes |
| $\alpha$ Reduction | SBB* | $1 - \frac{1}{k}$ | No | No |
| MIDA | SBB | $1 - \sqrt{\frac{ln(k)}{k}}$ | Yes | No |
| SBBA | SBB | $1 - \frac{1}{k}$ | Yes | No |
| * in expectation | | | | |

Table 3.14: Trade Reduction Mechanisms

Each of the trade reduction mechanisms in Table 3.14 satisfy IR, meaning traders are guaranteed non-negative utility, and IC, ensuring truthful bid reporting. Segal-Halevi et al., authors of the SBBA mechanism [87], highlight a gap in literature: The absence of a trade reduction mechanism that simultaneously achieves strong budget balance and determinism.

The next chapter begins by addressing this gap by presenting an impossibility theorem in Section 4.1.1, tackling the first of the three central research questions of this dissertation:

1. Is there a normalized, prior-free, and asymptotically efficient trade-reduction mechanism with a bid monotonic allocation rule that achieves both strong budget balance (SBB) and is deterministic?

The discussion of Strongly Budget-Balanced Double Auction (SBBA) mechanisms highlights the challenges of balancing efficiency and revenue generation, particularly in markets saturated by buyers or sellers. Existing mechanisms lack resiliency, in terms of maintaining efficiency, when faced with market saturation. This is a concern when

applying a trade reduction mechanism in unpredictable markets, such as those in emerging technology.

Through the analysis of various mechanisms, this chapter leads to a key insight that motivates the second research question. The discussion of Strongly Budget-Balanced Double Auction (SBBA) mechanisms highlights the challenge of balancing efficiency and revenue generation, particularly in markets that are saturated by either buyers or sellers. Current mechanisms lack resilience in maintaining efficiency when faced with market saturation, which is a significant concern when applying trade reduction mechanisms in unknown markets, as is the case in emerging technology. Chapter 4 identifies a source of this performance variability and provides a solution in Section 4.2, addressing the second research question of this dissertation.

2. Can a trade reduction mechanism be designed that holds the economic properties of IR, IC, and BB that is resilient to market saturation in terms of efficiency performance?

Table 3.14 highlights that only the McAfee and $\alpha$ Reduction mechanisms generate revenue. The $\alpha$ Reduction mechanism uses the hyperparameter $\alpha$ to alternate between the budget deficit VCG mechanism and the budget surplus basic trade reduction mechanism, a derivative of McAfee. In McAfee's mechanism, revenue is generated only through trade reduction, as clearing the market with ideal efficiency, using a single price, does not produce revenue.

Due to the randomization in the $\alpha$ Reduction mechanisms, the only mechanism guaranteed to produce a revenue in the event two or more efficient trades exists ($k > 1$) is the basic trade reduction. Because this mechanism is unable to clear the market with ideal efficiency, this leaves room for improvement in efficiency and motivates the third and final research question for this dissertation.

3. Can an IR, IC, and BB trade reduction mechanism be designed to guarantee a non-negative revenue while also achieving asymptotic efficiency?

The next chapter address this question in Section 4.3 through the use of hyper parameters that can be fine tuned to balance the trade-off between revenue and efficiency.

The three research questions will guide the presentation of new mechanisms that highlight the contributions of this dissertation and seek to overcome the limitations of existing designs. The incorporation of pre-filters and hyper parameters aim to **optimize** both efficiency and revenue in varying market conditions.

# 4

# NOVEL EXTENSIONS TO TR MECHANISMS

This chapter answers the three motivating research questions and presents the four primary research contributions.

The three research questions are:

1. Is there a normalized, prior-free, and asymptotically efficient trade-reduction mechanism with a bid monotonic allocation rule that achieves both strong budget balance (SBB) and is deterministic?
2. Can a trade reduction mechanism be designed that holds the economic properties of IR, IC, and BB that is resilient to market saturation in terms of efficiency performance?
3. Can an IR, IC, and BB trade reduction mechanism be designed to guarantee a non-negative revenue while also achieving asymptotic efficiency?

Section 4.1 addresses the first research question by presenting an impossibility theorem that highlights the necessary trade-off between determinism and strong budget balance. It then explores common pitfalls in mechanism design, particularly the challenge of maintaining efficient performance in markets saturated by either buyers or sellers.

A current challenge in mechanism design is achieving high efficiency without knowing prior distributions or market saturation, while avoiding the significant budget deficit typical of the VCG mechanism. Section 4.1.3 introduces a mechanism to overcome this challenge and represents the first research contribution: the design of a new mechanism that offers an alternative approach to overcoming the Myerson and Satterthwaite impossibility result [70], providing a robust alternative to the VCG mechanism. The mechanism achieves high efficiency regardless of market saturation with a revenue near zero. Because the mechanism produces either a small positive or negative revenue, it does not hold the economic property of budget balance.

Sections 4.2 and 4.3 address the second and third research questions, respectively. Section 4.4 then integrates the contributions from these two sections into a single mechanism.

These sections each introduce the design of a new mechanism and is organized by first describing the mechanism, then presenting proofs of the economic properties, and then finally analyzing the mechanisms efficiency and revenue performance in simulated markets. Performance is compared to existing mechanisms presented in Chapter 3.

## 4.1 MECHANISM DESIGN PITFALLS

The Myerson and Satterthwaite impossibility result [70] prohibits any mechanism from achieving all four economic properties of IR, IC, ideal efficiency, and budget balance simultaneously. careful consideration must be taken when designing mechanisms to achieve the desired economic properties to avoid pitfalls.

This section begins with the introduction of an impossibility theorem that answers the first research question by stating the necessary trade-off between determinism and strong budget balance. The section continues by showing the challenge of designing a mechanism that is IC.

In Section 3.4.3, market saturation (in which one side of the market represents a majority of traders) is presented as another mechanism design challenge. This motivates the second research question.

The final subsection explores an alternative approach to the circumvention of the Myerson and Satterthwaite impossibility result. A new mechanism is presented that sacrifices both budget balance and ideal efficiency, in favor of optimizing both. This new mechanism improves on the efficiency performance of the baseline McAfee and SBBA mechanism and provides and alternative to the VCG mechanism. This mechanism is able to minimize the revenue lost from the VCG mechanism.

### 4.1.1 *Impossibility Theorem*

The first research question that motivates this research is:

1. Is there a normalized, prior-free, and asymptotically efficient trade-reduction mechanism with a bid monotonic allocation rule that achieves both strong budget balance (SBB) and is deterministic?

This section presents an impossibility theorem that results in a necessary trade-off between SBB and determinism when a mechanism designer is under the following constraints.

The TR mechanism must:

- Be able to operate in the prior-free setting.
- Achieve the economic properties of IR, IC, and BB while being asymptotically efficient.
- Be normalized to ensure losing bidders do not incur a payment, which is a prerequisite to individual rationality.

Performing a deterministic trade reduction requires that the same bidders be reduced when the buyer and seller bids are identical.

Both the McAfee and SBBA mechanisms are able to produce a SBB and deterministic outcome when the ideally efficient allocation is produced. The issue arises when the outcome requires a trade reduction. Implementing the McAfee trade reduction mechanism results in a deterministic outcome but not a SBB mechanism. The SBBA mechanisms are SBB but the outcome is not deterministic.

**Theorem 5** : *For bilateral trades, a non-discriminatory, normalized, and deterministic, trade reduction mechanism $X = (A, P)$ with economic properties IR and IC and k efficient trades can only be SBB when $s_k = b_k$.*

*Proof.* Theorem 1 states that for a normalized mechanism $X$ to be IC, it must have a bid monotonic allocation rule $A$, and winning traders must pay their critical value.

For a mechanism to be deterministic during a trade reduction, the allocation and payment rules must specify which traders are reduced, ensuring identical outcomes for identical buyer and seller bids. A mechanism undergoing a trade reduction allows for at most $k - 1$ trades. For the allocation rule to be bid monotonic, the most competitive bidders must be assigned to the winning allocation. Equation 4.1 bounds the critical values for buyers, $c_b$, and sellers, $c_s$.

$$b_1 \geq ... \geq b_{k-1} \geq c_b \geq b_k \geq s_k \geq c_s \geq s_{k-1} \geq ... \geq s_1 \tag{4.1}$$

For a non-discriminatory mechanism to ensure strong budget balance (SBB), a single price $p$ must be used, making the critical values for both buyers and sellers equal: $p = c_b = c_s$. This can only occur when the least buyer and seller bids are equal, $s_k = b_k$. □

This holds for any trade reduction size.

Changing the definition of an efficient trade to require a gains from trade such that $b_k > s_k$ would eliminate the possibility of designing a deterministic trade reduction mechanism that is SBB. The mechanism designer is left with the trade-off between determinism and achieving strong budget balance.

### 4.1.2 *Non-truthful Trade Reduction Mechanisms*

The core concept behind the trade reduction mechanisms lies in the interaction between the least favorable traders (LFT), $s_k, b_k$, and the most competitive inefficient traders, $s_{k+1}, b_{k+1}$.

The relationship between efficient and inefficient traders are shown in equations 2.12 and 2.13, repeated here.

$$b_1, ..., b_{k-1} \geq b_k \geq s_k \geq s_{k-1}, ..., s_1 \tag{2.12}$$

$$b_{k+1} < s_{k+1} \tag{2.13}$$

These bids can be arranged in four possible ways:

1. $s_k \leq b_{k+1} \leq b_k \leq s_{k+1}$
2. $s_k \leq b_{k+1} \leq s_{k+1} \leq b_k$
3. $b_{k+1} \leq s_k \leq s_{k+1} \leq b_k$
4. $b_{k+1} \leq s_k \leq b_k \leq s_{k+1}$

Achieving ideal efficiency requires the selection of a price that is both below the least favorable buyer's willingness to pay ($b_k$) and above the least favorable seller's asking price ($s_k$).

In the first three arrangements there exists a bid (circled below) that is guaranteed to be satisfactory ($\in [s_k, b_k]$) for all efficient traders on both sides of the market.

1. $s_k \leq \textcircled{$b_{k+1}$} \leq b_k \leq s_{k+1}$
2. $s_k \leq \textcircled{$b_{k+1}$} \leq \textcircled{$s_{k+1}$} \leq b_k$
3. $b_{k+1} \leq s_k \leq \textcircled{$s_{k+1}$} \leq b_k$
4. $b_{k+1} \leq s_k \leq b_k \leq s_{k+1}$

The fourth arrangement, however, lacks a bid that would guarantee an ideally efficient outcome. The famous Myerson and Satterthwaite impossibility result [70] comes from the absence of a guaranteed price point in the fourth arrangement.

The McAfee TR mechanism selects a price halfway between the most competitive inefficient traders, $s_{k+1}, b_{k+1}$. This mechanism is based on guaranteeing an ideally efficient allocation in the second arrangement, $s_k \leq b_{k+1} \leq p_0 \leq s_{k+1} \leq b_k$. In all other cases, clearing the market with an ideally efficient allocation is possible but not guaranteed.

The SBBA mechanisms are able to guarantee an ideally efficient allocation in multiple arrangements at the expense of a guaranteed trade reduction in the remaining cases. These mechanisms select a bid (rather than estimate a price).

The first SBBA mechanism uses equation 3.25 as an initial price to try and clear the market with ideal efficiency.

$$p_1 = \min(s_{k+1}, b_k) \tag{3.25}$$

Considering the four possible bid arrangements, the SBBA-1 mechanism results in ideal efficiency in arrangements two and three.

1. $s_k \leq b_{k+1} \leq b_k \leq s_{k+1}$
2. $s_k \leq b_{k+1} \leq \textcircled{s_{k+1}} \leq b_k$
3. $b_{k+1} \leq s_k \leq \textcircled{s_{k+1}} \leq b_k$
4. $b_{k+1} \leq s_k \leq b_k \leq s_{k+1}$

Similarly, the price used in the second SBBA mechanism, originally shown in equation 3.26, results in the same outcomes as the generalized price from equation 4.8 when setting epsilon to one, $\epsilon = 1$.

$$p_2 = \max(s_k, b_{k+1}) \tag{3.26}$$

1. $s_k \leq \textcircled{b_{k+1}} \leq b_k \leq s_{k+1}$
2. $s_k \leq \textcircled{b_{k+1}} \leq s_{k+1} \leq b_k$
3. $b_{k+1} \leq s_k \leq s_{k+1} \leq b_k$
4. $b_{k+1} \leq s_k \leq b_k \leq s_{k+1}$

Combining the McAfee price estimate, $p_0 = \frac{1}{2}(b_{k+1} + s_{k+1})$ with either of the SBBA trade reduction methods would result in a loss of incentive compatibility. Should a random buyer be reduced, as in the second SBBA mechanism, then buyer $k$ would have an incentive to strategically alter their bid. Even if buyer $k$ would be part of the winning allocation, $b_k \geq p_0 \geq s_k$, they would still have an incentive to reduce their bid and force a trade reduction in order to trade at a lower price, $b_{k+1}$. The larger the market, they less likely $b_k$ would be randomly reduced.

A mechanism designer must carefully a select a price that is likely to result in an ideally efficient. The pairing of an allocation and payment rule are crucial to maintaining truthfulness. For example, a mechanism designer may notice that the first three bid arrangements all contain at least one inefficient bid that could be used to select a price.

An approach would be to use both the most competitive bids inefficient bids $b_{k+1}$ and $s_{k+1}$ to a design a trade reduction mechanism, shown in equation 4.2.

$$X_{BandS}(A,P) : (\mathbf{b},\mathbf{s}) \rightarrow \begin{cases} (\mathbf{\Gamma}, \{s_{k+1}, b_{k+1}\}) & s_{k+1}, b_{k+1} \in [s_k, b_k] \\ (\mathbf{\Gamma}, \{s_{k+1}, s_{k+1}\}) & s_{k+1} \in [s_k, b_k], b_{k+1} \notin [s_k, b_k] \\ (\mathbf{\Gamma}, \{b_{k+1}, b_{k+1}\}) & b_{k+1} \in [s_k, b_k], s_{k+1} \notin [s_k, b_k] \\ (\mathbf{\Gamma} - \{b_k, s_k\}, \{b_k, s_k\}) & s_{k+1}, b_{k+1} \notin [s_k, b_k] \end{cases} \tag{4.2}$$

The use of both prices only occurs when both of the most competitive bids inefficient bids $b_{k+1}$ and $s_{k+1}$ are satisfactory to each side of the market. This only occurs in arrangement two. Using both prices require buyers to pay the higher price, $s_{k+1}$, while sellers receive the lower, $s_{k+1}$ to ensure the mechanism is budget balanced. In this case however, the least favorable buyer, $b_k$, could strategically determine to report a bid that is lower than their true valuation price such that the most competitive inefficient seller bid, $s_{k+1}$ is no longer acceptable resulting in bid arrangement one. Buyer $k$ is thus able to reduce the price for all buyers, increasing their utility. Seller $k$ would be able to achieve the same. This mechanism would loose the incentive compatibility property.

The use of a single price, should bids fall according to arrangement two, only solves this problem for one side of the market. For example should the mechanism designer determine bid $s_{k+1}$ to be the trump bid, then buyer $k$ would still be incentivized to strategically alter their bid to change the bid arrangements to reflect arrangement one, in which case, all buyers would pay a lower price.

Reducing the trade should bids fall into arrangement two does not solve the problem either. Such a mechanism, shown in equation 4.3, would use either the most competitive inefficient buyer bid, $b_{k+1}$, or seller bid, $s_{k+1}$, but not both.

$$X_{BORS}(A,P) : (\mathbf{b},\mathbf{s}) \rightarrow \begin{cases} (\mathbf{\Gamma} - \{b_k, s_k\}, \{b_k, s_k\}) & s_{k+1}, b_{k+1} \in [s_k, b_k] \\ (\mathbf{\Gamma}, \{s_{k+1}, s_{k+1}\}) & s_{k+1} \in [s_k, b_k], b_{k+1} \notin [s_k, b_k] \\ (\mathbf{\Gamma}, \{b_{k+1}, b_{k+1}\}) & b_{k+1} \in [s_k, b_k], s_{k+1} \notin [s_k, b_k] \\ (\mathbf{\Gamma} - \{b_k, s_k\}, \{b_k, s_k\}) & s_{k+1}, b_{k+1} \notin [s_k, b_k] \end{cases} \tag{4.3}$$

In this mechanism buyer $k$ is incentived to strategically misrepresent their bid as well. By strategically reporting a bid less than their true valuation such the bid $s_{k+1}$ is no longer acceptable would result in an ideally efficient allocation. Buyer $k$ can enter into the winning allocation and gain a non-negative utility by under-reporting their true valuation.

Theorem 1 states that for a mechanism to be IC each side of the market must have a **single** critical value that determines the winning allocation. The trade reduction introduces a second critical value for each side of the market, violating incentive compatibility.

### 4.1.3 *Alternative Mechanism to VCG*

The Myerson and Satterthwaite impossibility result [70] states that no mechanism can achieve all four economic properties of IR, IC, BB, and ideal efficiency. The McAfee mechanism [60] is able to circumvent this by sacrificing ideal efficiency. The mechanism can achieve ideal efficiency but does not guarantee it. When efficiency must be sacrificed, a trade reduction of the least favorable trade minimizes the loss that becomes negligible as the market grows.

Equation 4.4 uses the trade reduction concept to introduce an alternate mechanism to VCG, $X_{VCG-alt}$.

$$X_{VCG-alt}(A,P) : (\mathbf{b},\mathbf{s}) \rightarrow \begin{cases} (\mathbf{\Gamma}, \{b_{k+1}, s_{k+1}\}) & s_{k+1}, b_{k+1} \in [s_k, b_k] \\ (\mathbf{\Gamma}, \{s_{k+1}, s_{k+1}\}) & s_{k+1} \in [s_k, b_k], b_{k+1} \notin [s_k, b_k] \\ (\mathbf{\Gamma}, \{b_{k+1}, b_{k+1}\}) & b_{k+1} \in [s_k, b_k], s_{k+1} \notin [s_k, b_k] \\ (\mathbf{\Gamma} - \{b_k, s_k\}, \{b_k, s_k\}) & s_{k+1}, b_{k+1} \notin [s_k, b_k] \end{cases} \tag{4.4}$$

This mechanism is able to clear the market with ideal efficiency in three of the four possible bid arrangements. When both of the most competitive inefficient bids are acceptable price points for each side of the market, $s_{k+1}, b_{k+1} \in [s_k, b_k]$, the market clears with ideal efficiency. The selected prices, $p_b = b_{k+1}, p_s = s_{k+1}$, however, cause a budget deficit ($p_b - p_s < 0$) because traders at $k+1$ are inefficient ($b_{k+1} < s_{k+1}$).

A single price is used when either the most competitive inefficient buyer or seller is an acceptable price point. This results in a budget of exactly zero. Otherwise a trade reduction is performed according to McAfee resulting in a non-negative revenue. This counteracts the deficit lost while increasing the efficiency relative to previous TR mechanisms.

**Lemma 10** : *The alternate VCG mechanism, $X_{VCG-alt}$, satisfies the property of Individual Rationality.*

*Proof.* By considering the buyers side of the market, when $b_{k+1} \in [s_k, b_k]$ all efficient buyers trade at $p_b = b_{k+1}$. By equation 2.10 ($b_1 \geq b_2 \geq ... \geq b_n$) buyers will always pay less than their bid. Under the condition $s_{k+1} \in [s_k, b_k], b_{k+1} \notin [s_k, b_k]$ Buyers pay $s_{k+1}$ which by the definition of the condition is less then their bid.

Finally when a trade reduction is performed, buyer $b_k$ is removed from trading. Equation 2.10 states $b_{k-1}$ and all subsequent trading buyers are more competitive than the least favorable buyer: $(b_1, ..., b_{k-1}) \geq b_k = p_b$, hereby satisfying IR. Arguments for the sellers side of the market are analogous. □

**Lemma 11** : *The alternate VCG mechanism, $X_{VCG-alt}$, satisfies the property of Incentive Compatibility.*

*Proof.* All losing traders do not pay or receive anything satisfying Definition 37 for a normalized mechanism. Given a trader is in the final allocation, a trader with a more competitive bid will also be in the final allocation. According to Definition 38 this allocation rule is bid monotonic.

Considering the buyers side of the market, under the condition $s_k \leq b_{k+1}$ all buyers trade and pay $b_{k+1}$. Under the condition $s_k > b_{k+1}$ all buyers above their critical value of $c_i = s_{k+1}$ trade and pay their critical value. Under all other conditions the least favorable traders (LFT) are reduced. All buyers above their critical value $c_i = b_k$ trade and pay their critical value. Arguments follow analogous for the seller's side. By theorem 1, $X_{VCG-alt}$ is IC. □

**Theorem 6** : *The alternate VCG mechanism, $X_{VCG-alt}$, is prior-free, deterministic, and satisfies the properties of IR and IC while achieving asymptotic efficiency.*

*Proof.* By Definition 43 the alternate VCG mechanism is deterministic because it does not insert randomization into any part of the mechanism. The rules of the mechanism are not dependent on the prior distributions of the buyer's valuation or seller's reservation prices and is able to operate "prior-free" satisfying Definition 42. Lemma 10 proves the mechanism is IR and Lemma 11 proves the mechanism is IC. At most one trade is reduced from the efficient allocation causing $X_{VCG-alt}$ to converge at a rate $1 - \frac{1}{k}$ proving the mechanism is asymptotically efficient. □

The alternate VCG mechanism's performance in terms of efficiency and revenue can be compared against existing trade reduction mechanisms. Figure 4.1 shows normalized efficiency as the market size grows. Each market size is simulated 10,000 times drawing buyer's and seller's bids from a normalized uniform distribution. Traders are equally likely to enter the market as a buyer or seller.

The alternate VCG mechanism, $X_{VCG-alt}$, is able to improve upon the asymptotic efficiency of both the McAfee and SBBA mechanisms. Figure 4.2 shows the revenue for the same markets.

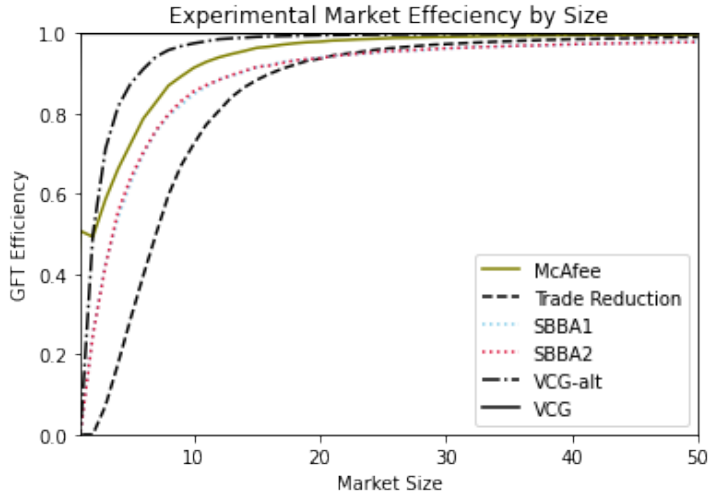



Figure 4.1: Efficiency for VCG-alt mechanism

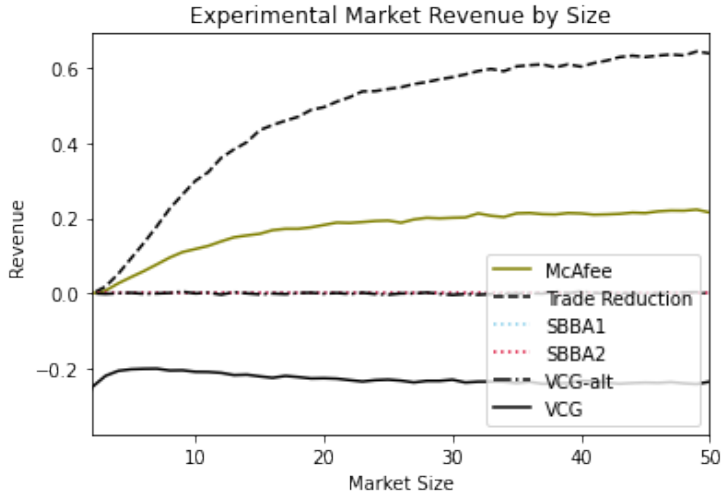



Figure 4.2: Revenue for VCG-alt mechanism

While the alternate VCG mechanism does not guarantee a non-negative budget, the average revenue it produces is near zero. Unlike the VCG mechanism, the alternate VCG mechanism can result in a positive revenue. By sacrificing a fraction of efficiency this mechanism is able to reduce the cost to implement an individually rational and Incentive compatible mechanism at a fraction of the cost.

The deterministic alternative VCG mechanism is able to further separate itself from existing trade reduction mechanisms in buyer or seller saturated markets. Figure 4.3 shows the efficiency performance in a buyer saturated market.

The difference in efficiency between the alternate VCG mechanism and the most efficient budget balanced mechanism is much smaller in a saturated market. This occurs because the SBBA-2 mechanism is extremely efficient in buyer saturated markets. Both the SBBA-2 mechanism and the alternate VCG mechanism are more efficient than other mechanisms.

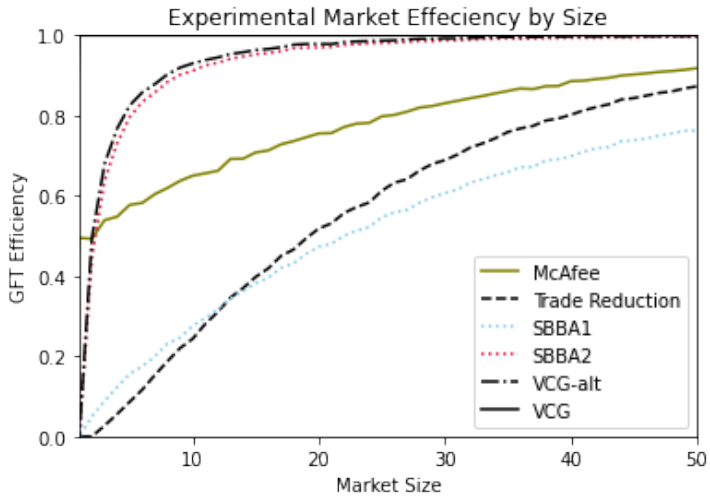



Figure 4.3: Efficiency (buyer saturated market)

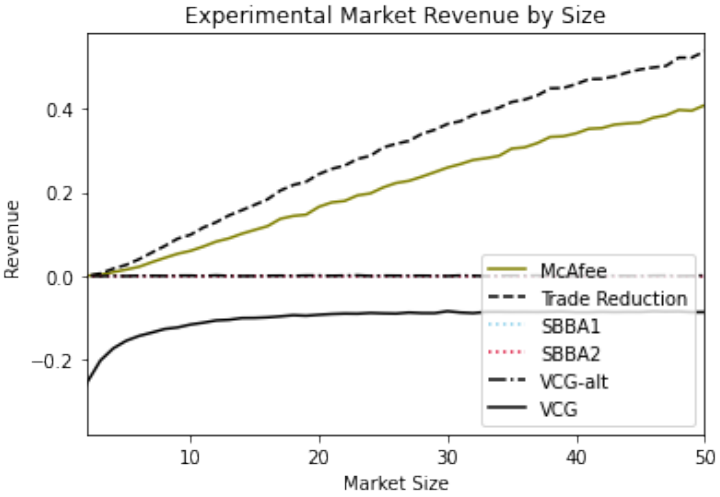



Figure 4.4: Revenue (buyer saturated market)

Figure 4.4 shows the alternate VCG mechanism with a revenue near zero. Whether the mechanism produces a small positive or negative revenue depends on the bid arrangements of that specific market. The average revenue is near zero, $\Pi_{X_{VCG-alt}} = 0 \pm 0.001$ in each market size.

What separates the alternate VCG mechanism from existing mechanisms is the resilient performance regardless of market saturation. Figure 4.5 shows a seller saturated market. In these markets the alternate VCG mechanism performance similar to, though slightly better than, the SBBA-1 mechanism.

If knowledge of prior distributions and market saturation in unknown, the selection of an existing mechanism is risky. The difference between the efficiency achieved by the alternate VCG mechanism and the SBBA-2 mechanism in Figure 4.5 is significant. The efficiency of both SBBA mechanisms depend largely on the market saturation. Figure 4.6 shows the same near zero revenue.

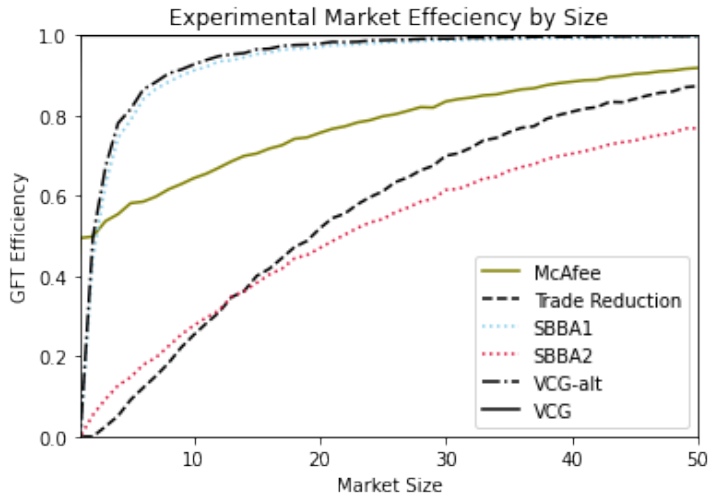



Figure 4.5: Efficiency (seller saturated market)

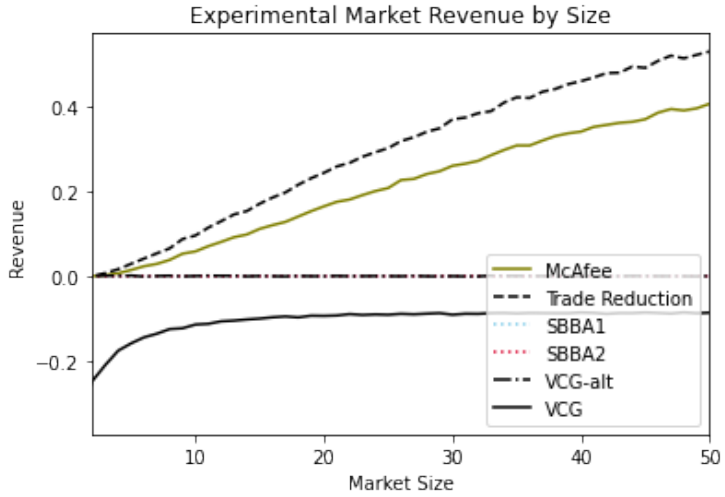



Figure 4.6: Revenue (seller saturated market)

Comparing the efficiency performance in both even and saturated markets in Figures 4.1, 4.3, and 4.5 shows the alternate VCG mechanism as the most efficient. The revenue in Figures 4.2, 4.4, and 4.6 show the mechanism is near zero independent of saturation. This mechanism is resilient to any market makeup.

Since at least one of the four desired economic properties (IR, IC, BB, and ideal efficiency) must be sacrificed, the alternative VCG mechanism chooses to forgo both BB and efficiency. This approach, similar to the McAfee mechanism, bypasses the impossibility theorem, maximizing efficiency while minimizing revenue loss. As a result, the newly introduced "alternate VCG" mechanism is able to achieve ideal efficiency and also able generate positive revenue. While the mechanism is able to achieve both in a single outcome, the mechanism is not guaranteed to achieve either and therefore does not satisfy the definitions for the economic properties of either ideal efficiency or budget balance.

## 4.2 EXTENSION 1: A PRE-FILTER

The alternate VCG mechanism, introduced in Section 4.1.3, offers a resilient solution to market saturation. Unlike the SBBA mechanisms, which perform well under saturation on one side of the market but not the other, the alternate VCG mechanism can slightly improve efficiency regardless of whether the market is buyer- or seller-saturated. If market saturation is unknown before implementation, choosing the wrong SBBA mechanism could lead to a significant reduction in efficiency.

The limitation with the newly designed alternate VCG mechanism is the inability to achieve the economic property of budget balance. This property can prove essential in mechanism design to ensure financial viability. While the alternate VCG mechanism roughly results in neither a surplus nor deficit, it does not provide a guarantee of either. Budget balance incentives the market itself allowing the mechanism to be financially sustainable and self-sufficient. Achieving this property can also prove necessary to comply with legal or institutional constraints.

The main limitation of the SBBA mechanism in saturated markets is that the under-saturated side tends to be fully consumed by efficient trades. In other words, all traders on the under-saturated side become efficient.

The McAfee mechanism is able to mitigate this lack of inefficient traders with fictitious order statistics [60]. If an inefficient buyer does not exists one can be created with a value of zero on a normalized scale to set the lower boundary. Similarly an inefficient can be created with a value of one on a normalized scale to mark the upper bound. This allows the McAfee mechanism to estimate a price half-way between these values to try and clear the market with ideal efficiency.

The use of fictitious order statistics is ineffective for the SBBA pricing rules which rely on the existence of an inefficient buyer or seller to clear the market with ideal efficiency. For example, the first SBBA mechanism, $X_{SBBA1}$, with pricing rule $p_1 = \min(s_{k+1}, b_k)$ struggles in buyer saturated markets as all sellers are often efficient. The use of a fictitious order statistic, $s_{k+1} = 1$, would result in a trade reduction and a clearing price of $p_1 = b_k$.

The four possible bid arrangements between the least favorable traders $k$ and most competitive inefficient traders $k+1$ are:

1. $s_k \leq b_{k+1} \leq b_k \leq s_{k+1}$
2. $s_k \leq b_{k+1} \leq s_{k+1} \leq b_k$
3. $b_{k+1} \leq s_k \leq s_{k+1} \leq b_k$
4. $b_{k+1} \leq s_k \leq b_k \leq s_{k+1}$

A buyer saturated market often results in the need for a fictitious seller to set the upper bound.

The increased number of buyers makes it more likely the most competitive inefficient buyer, $b_{k+1}$, would also be willing to trade with the least favorable efficient seller, $s_k \leq b_{k+1}$. This results in the first bid arrangement being the most likely outcome. Rather than reducing the LFT trade due to the lack of an inefficient seller (when all sellers are efficient), a desirable outcome would be to clear the market with a price equal to $b_{k+1}$. This would overcome the efficiency lost from a trade reduction. The second SBBA mechanism has similar struggles in seller saturated markets as an inefficient buyer often does not exist.

### 4.2.1 *Pre-Filter Description*

The first SBBA mechanism requires a check for whether all sellers are efficient. This check can be accomplished through a pre-filter that compares the number of efficient traders $k$ with the total number of buyers $n$ and sellers $m$. If all sellers are efficient such that $k = m$, the mechanism should attempt to clear the market with ideal efficiency, using the most competitive buyer bid, $b_{k+1}$, as a price estimate.

$$X_{SBBA1-filter}(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow \begin{cases} (\mathbf{\Gamma}, \{b_{k+1}, b_{k+1}\}) & k = m \ \& \ b_{k+1} \in [s_k, b_k] \\ (\mathbf{\Gamma} - \{b_R, s_k\}, \{s_k, s_k\}) & k = m \ \& \ b_{k+1} \notin [s_k, b_k] \\ (\mathbf{\Gamma}, \{s_{k+1}, s_{k+1}\}) & k \neq m \ \& \ s_{k+1} \leq b_k \\ (\mathbf{\Gamma} - \{b_k, s_R\}, \{b_k, b_k\}) & \text{Otherwise} \end{cases} \tag{4.5}$$

Equation 4.5 shows the use of the pre-filter. When an inefficient seller does not exist ($k = m$), the mechanism operates as the SBBA-2 mechanism normally would. The mechanism uses the most competitive buyer bid as a single price estimate to attempt to clear the market with ideal efficiency, $b_{k+1} \in [s_k, b_k]$. If a trade reduction is performed a random buyer must be reduced to maintain the properties of IC and SBB. When an inefficient seller exists ($k \neq m$), the pre-filter does not apply, and the SBBA-1 operates as normal.

Extending the SBBA-1 mechanism to include a pre-filter does not change the economic properties of the mechanism.

**Corollary 1** : *$X_{SBBA1-filter}$ is prior-free, deterministic, and satisfies the properties of IR, IC, and SBB while achieving asymptotic efficiency.*

*Proof.* If an inefficient seller does not exist, the number of efficient trades cannot be increased. Inefficient traders cannot decrease the number of efficient trades by strategically manipulating their bid. An efficient trader can only reduce the number of efficient trades by removing themselves from the efficient allocation, which cannot increase their utility.

If an inefficient seller does exist, lowering their bid to make all sellers efficient decreases the price. The inefficient seller either trades at $b_{k+1}$ or $s_k$ and incurs negative utility or is reduced from trading. In both scenarios, the inefficient seller is not incentivized to misrepresent their reservation price. By Definition 31, an inefficient buyer who raises their bid to make all sellers efficient would trigger a trade reduction, as $b_{k+2} \leq b_{k+1}$. This results in the random reduction of a buyer and seller $s_{k+1}$, which is used to clear the market. The buyer is either reduced leading to a utility of zero, or trades at $s_{k+1}$ leading to negative utility. Either way an inefficient cannot increase their utility.

Therefore, traders cannot improve their utility by influencing the application of the pre-filter. When all sellers are efficient ($k = m$), the mechanism functions identically to the SBBA-2 mechanism. When an inefficient seller exists, the mechanism behaves like the SBBA-1 mechanism. All property proofs follow directly from those established for Theorem 4. □

Equation 4.6 shows how a similar filter can be applied to the SBBA-2 mechanism.

$$X_{SBBA2-filter}(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow \begin{cases} (\mathbf{\Gamma}, \{s_{k+1}, s_{k+1}\}) & k = n \ \& \ s_{k+1} \in [s_k, b_k] \\ (\mathbf{\Gamma} - \{b_k, s_R\}, \{b_k, b_k\}) & k = n \ \& \ s_{k+1} \notin [s_k, b_k] \\ (\mathbf{\Gamma}, \{b_{k+1}, b_{k+1}\}) & \min\{n, m\} = n \neq k \ \& \ b_{k+1} \geq s_k \\ (\mathbf{\Gamma} - \{b_R, s_k\}, \{s_k, s_k\}) & \text{Otherwise} \end{cases} \tag{4.6}$$

The same arguments and proofs apply to $X_{SBBA2-filter}$, giving it identical properties to $X_{SBBA1-filter}$. When implemented in a saturated market for which it was not designed, each SBBA mechanism experiences a loss in efficiency, primarily due to the absence of inefficient traders. The use of pre-filters addresses this inefficiency, resulting in a more resilient mechanism. With the pre-filter, both $X_{SBBA1-filter}$ and $X_{SBBA2-filter}$ maintain high efficiency, regardless of market saturation.

### 4.2.2 *Efficiency Performance*

Mechanisms $X_{SBBA1-filter}$ and $X_{SBBA2-filter}$ are strongly budget balanced and achieve a revenue of exactly zero in each and every market. Efficiency performance of these two new mechanisms are graphed along side the original SBBA mechanisms. The McAfee, basic trade reduction, and VCG mechanisms are shown to provide baseline performance. Each market size is simulated 10,000 times with buyers and sellers being drawn from normalized uniform distributions. Figure 4.7 shows the efficiency performance in balanced markets where traders are equally likely to enter the market as a buyer or as a seller.

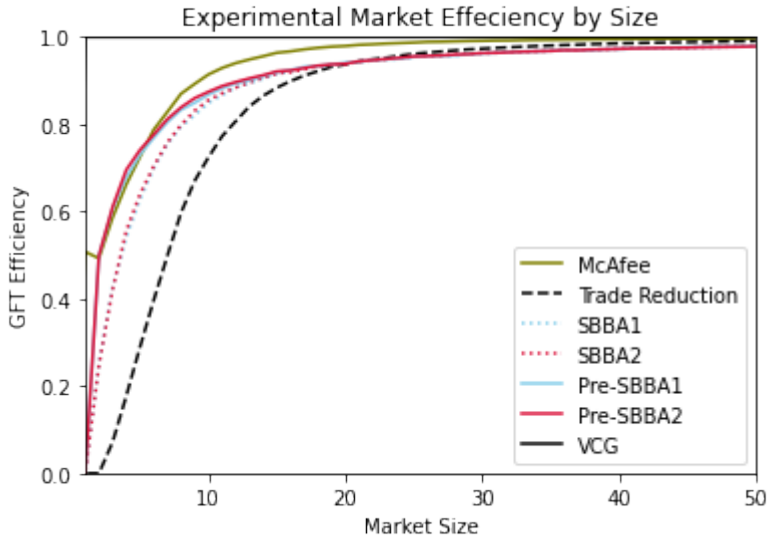



Figure 4.7: Pre-filter with SBBA

Adding the pre-filter improves SBBA efficiency in smaller market sizes (less than 10) in the non-saturated markets shown in Figure 4.7. As the market grows, the likelihood that all traders are efficient decreases. The existence of an inefficient trader become more and more likely. This reduces the impact of the pre-filter. Because of this, Figure 4.7 shows the SBBA mechanisms with a pre-filter converging to the original SBBA mechanisms.

Additionally, the basic trade reduction mechanism eventually surpasses the SBBA mechanisms in efficiency. This is because the SBBA mechanisms randomly reduce one side of the market during a trade reduction, whereas the basic trade reduction mechanism specifically reduces the least favorable trade.

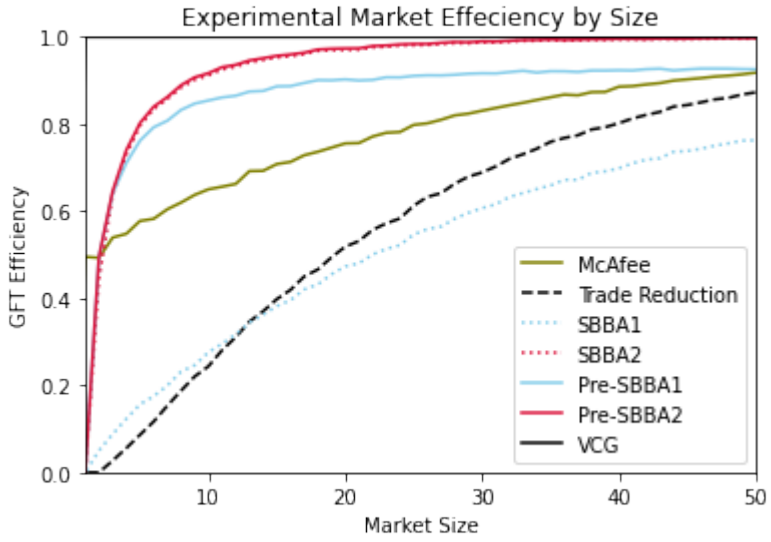



Figure 4.8: Pre-filter with SBBA buyer saturation

Figure 4.8 shows the same mechanisms in buyer saturated markets. In these markets a trader has a 90% chance of entering the market as a buyer. All traders continue to be drawn from a normalized uniform distribution. In the event an inefficient buyer or seller does not exist, fictitious order statistics are created to determine an initial price, $b_{k+1} = 0, s_{k+1} = 1$. This are extremely accurate determinations of the order statistics as they are the exact upper and lower bounds of the traders prior distributions. Further the initial price calculated by the McAfee mechanism, $p_0 = \frac{1}{2}(b_{k+1} + s_{k+1})$, results in a reasonably good estimate to attempt to clear the market with ideal efficiency.

If the upper bound was not known, and a fictitious order statistic was determined to be equal to two or even ten times the expected upper boundary, the McAfee mechanism would not result in many ideally efficient allocations. Inaccurate determinations of the order statistics would result in efficiency performance similar to that of the basic trade reduction mechanism.

Performance in seller saturated markets, shown in Figure 4.9, is similar.

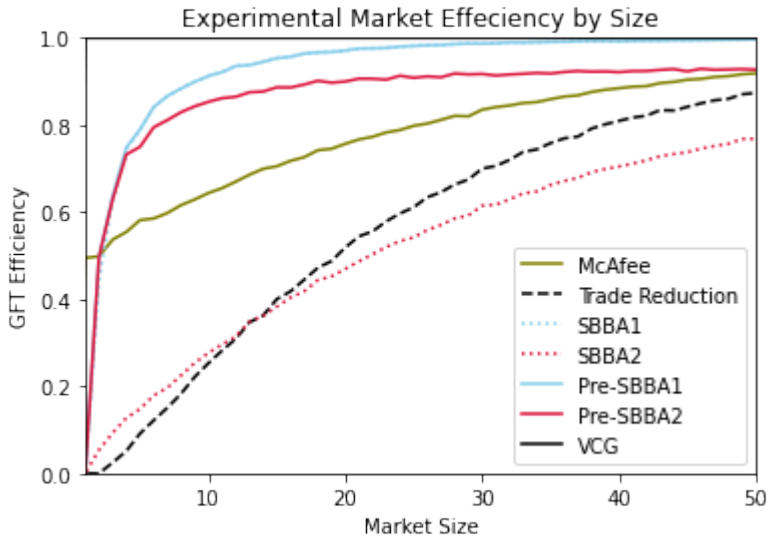



Figure 4.9: Pre-filter with SBBA seller saturation

In seller saturated markets the SBBA-1 mechanism is much more efficient then the SBBA-2 mechanism. A seller saturated market also means that a pre-filter on the SBBA-1 mechanism has very little effect as the market almost always has an inefficient seller. The existence of an inefficient seller means the $X_{SBBA1-filter}$ acts identical to the original mechanism, $X_{SBBA1}$.

A pre-filter on the SBBA-2 mechanism has a significant improvement in efficiency over the original SBBA-2 mechanism, seen in Figure 4.9. In a seller saturated market all buyers are often efficient which allows the pre-filter to use the most competitive inefficient seller as a price estimate to try and clear the market with ideal efficiency. The use of a pre-filter allows $X_{SBBA2-filter}$ to achieve similar, but slightly lower efficiency than the original SBBA-1 mechanism.

Similar arguments can be made for $X_{SBBA1-filter}$ in a buyer saturated market shown in Figure 4.8.

Both the $X_{SBBA1-filter}$ and $X_{SBBA2-filter}$ are able to achieve high efficiency in all markets regardless of saturation by one side of the market, which address the second research questions.

2. Can a trade reduction mechanism be designed that holds the economic properties of IR, IC, and BB that is resilient to market saturation in terms of efficiency performance?

These mechanisms able to maintain efficiency regardless of market saturation while maintaining the desired economic properties of IR, IC, and BB. The mechanisms are not deterministic; however, they do strengthen the budget balance property to *strong* budget balanced.

## 4.3 EXTENSION 2: HYPERPARAMETERS

This section presents a new mechanism that extends the McAfee trade reduction mechanism by incorporating two hyperparameters, epsilon and upsilon. To clarify this generalization, a simplified version of the mechanism is introduced that uses only one hyperparameter. This single-hyperparameter mechanism not only provides a foundation for understanding the full generalization but also serves as a direct extension of the original McAfee trade reduction mechanism [60].

### 4.3.1 *Single Hyperparameter Mechanism*

The initial price used the McAfee mechanism to attempt to clear the market with ideal efficiency is shown in equation 4.7.

$$p_0 = 0.5b_{k+1} + 0.5s_{k+1} \tag{4.7}$$

This equation can be generalized through the use of a hyperparameter, $\epsilon \in [0, 1]$, shown in equation 4.8.

$$p_\epsilon = \epsilon b_{k+1} + (1 - \epsilon)s_{k+1} \tag{4.8}$$

This creates a continuous space for a price estimate to be determined. The price can be selected according to any available information including prior distributions and/or market saturation. When $\epsilon = 0.5$ equation 4.8 reduces to the McAfee price $p_0$ in equation 3.3.

This generalization uses a single price to attempt to clear the market with ideal efficiency. Clearing the market with ideal efficiency would therefore result in a revenue of exactly zero, like the McAfee mechanism.

The full single-hyperparamter mechanism acts as a generalization of the McAfee mechanism and is shown in equation 4.9.

$$X_{gen-Mc}(A,P) : (\mathbf{b},\mathbf{s}) \rightarrow \begin{cases} (\mathbf{\Gamma}, \{p_\epsilon, p_\epsilon\}) & p_\epsilon \in [s_k, b_k] \\ (\mathbf{\Gamma} - \{b_k, s_k\}, \{b_k, s_k\}) & p_\epsilon \notin [s_k, b_k] \end{cases} \tag{4.9}$$

$X_{gen-Mc}$ allows for increased flexibility should additional market information, or knowledge of prior distribution be available. Like McAfee the generalized mechanism does not require any prior knowledge to be implemented. With no information the generalized mechanism can simply use $\epsilon = 0.5$ to reflect the baseline McAfee mechanism. Partial market information can be leveraged to determine a more optimal value of $\epsilon \in [0, 1]$.

In example, assuming maximizing efficiency is the goal, if the exact prior distribution along which buyer valuations fall remains unknown but the average, or perhaps the expected number of buyers is known to be larger than the expected number of sellers, $\epsilon$ can be set to one, $\epsilon = 1$, to reflect that information.

Setting $\epsilon = 0$ would reduce efficiency and result in more trade reductions. Consequently this would also **maximize revenue**. The generalized McAfee mechanism, $X_{gen-Mc}$, gives the mechanism designer the option of selecting an estimate that best represents their understanding of the market while not requiring any information be available to be used.

This generalized Mcafee mechanism naturally maintains the same economic properties of IR, IC, BB, and asymptotic efficiency.

**Corollary 2** : *The generalized McAfee mechanism, $X_{gen-Mc}$, is prior-free, deterministic, and satisfies the properties of IR, BB, and IC while achieving asymptotic efficiency.*

*Proof.* Given that the trade reduction is performed identically to that of the McAfee mechanism, all property proofs follow analogously from the proofs laid out for theorem 2, as they only depend on the relative position of $p_\epsilon$ compared to the bids:

$$b_1 > ...b_k > p_\epsilon > s_k > ... > s_1 \tag{4.10}$$

□

While efficiency is an obvious measure of mechanism success, markets may wish to produce revenue as a secondary objective. An outcome in which the mechanism produces a revenue of zero becomes less desirable.

The generalized McAfee mechanism only produces a revenue in the event a trade reduction is performed. A mechanism designer may choose to not consider a price that could result in an ideally efficient allocation and elect to implement the basic trade reduction mechanism from equation 3.6. This forgoes the possibility of an ideally efficient outcome and always reduces the least favorable trade (LFT). This approach **maximizes revenue** while maintaining asymptotic efficiency.

$$X_{TR}(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow (\mathbf{\Gamma} - \{b_k, s_k\}, \{b_k, s_k\}) \tag{3.6}$$

The use of a second hyperparameter, one for each side of the market, would allow a mechanism to both achieve ideal efficiency and produce a revenue to cover the operating costs of the market.

### 4.3.2 *Multi-Hyperparameter Mechanism*

Both the McAfee mechanism and the Generalized McAfee result in one of three outcomes.

1. An ideally efficient outcome with a revenue of zero:
   $p_b = p_s = p_0, p_\epsilon$.
2. The LFT trade is reduced and their bids are used to generate a positive revenue:
   $p_b = b_k > s_k = p_s$.
3. The LFT trade is reduced and their equal bids result in a revenue of zero:
   $p_b = b_k = s_k = p_s$.

When $b_k = s_k$, there is no possible way to produce a revenue without a change in definition of efficient traders. Definition 20 states that a trade between a buyer and seller is efficient if it results in a non-negative utility, $b_k \geq s_k$. The case when bids are equal, $b_k = s_k$, is therefore considered efficient even though this trade produces zero gains.

If the mechanism designer wishes to produce a revenue in every outcome where the buyer bid is strictly larger than the seller bid, $b_k > s_k$, they are left with a single option: the basic trade reduction mechanism in which the LFT is immediately reduced. While beyond the scope of this research, a change in the definition of an efficient trade to require the buyer bid be strictly larger than the seller, $b_k > s_k$, would avoid this third outcome at no cost to the mechanisms overall GFT.

The upsilon reduction mechanism, $X_v$, offers and alternative to the basic trade reduction mechanism using a second hyperparameter upsilon, $v \in [0, 1]$. The use of multiple hyperparameters allows for separate price estimates for buyers ($p_b$) and sellers ($p_s$). This shifts the single price point estimate to a range of values, $[p_s, p_b]$, which must be accepted by both sides of the market to achieve an ideally efficient allocation, $[p_s, p_b] \in [s_k, b_k]$.

This new mechanism uses a second hyperparameter to calculate separate prices, allowing the market to be cleared with ideal efficiency while simultaneously generating revenue. Should the mechanism designer wish to change the definition of an efficient trade, the upsilon reduction mechanism, $X_\upsilon$, would guarantee a positive revenue in every outcome that clears at least one trade.

Like the generalized McAfee mechanism, $X_{gen-Mc}$ a hyperparameter is introduced. The generalized price, $p_\epsilon$, in equation 4.8, can be further manipulated to produce additional revenue by splitting the generalized price into separate prices for each side of the market.

The price sellers receive is denoted $p^\epsilon$ according to equation 4.11 while the price buyers pay, $p^\upsilon$, is calculated with the additional hyperparameter in equation 4.12.

$$p_s = p^\epsilon = \epsilon(b_{k+1}) + (1-\epsilon) * (s_{k+1}) \tag{4.11}$$
$$p_b = p^\upsilon = \upsilon(b_{k+1}) + (1-\upsilon) * (s_{k+1}) \tag{4.12}$$

To ensure the mechanism is BB the following constraint is applied:

$$\epsilon \geq \upsilon \tag{4.13}$$

Equation 4.13 forces the buyer's price to be greater than or equal to the seller's price.

$$p^\upsilon \geq p^\epsilon \tag{4.14}$$

The upsilon reduction mechanism, $X_\upsilon$, becomes

$$X_\upsilon(A, P) : (\mathbf{b}, \mathbf{s}) \rightarrow \begin{cases} (\mathbf{\Gamma}, \{p^\upsilon, p^\epsilon\}) & (p^\epsilon, p^\upsilon) \in [s_k, b_k] \\ (\mathbf{\Gamma} - \{b_k, s_k\}, \{b_k, s_k\}) & \text{Otherwise} \end{cases} \tag{4.15}$$

The upsilon reduction mechanism $X_\upsilon$ produces the ideally efficient outcome when both the buyer's and seller's prices fall into the acceptable range, $p^\upsilon, p^\epsilon \in [s_k, b_k]$. If either price is unacceptable a trade reduction is performed identical that introduced by McAfee. As $s_{k+1}$ and $b_{k+1}$ are inefficient such that $b_{k+1} < s_{k+1}$ by equation 2.13 the additional constraint on the hyperparameters $\epsilon$ and $\upsilon$ shown in equation 4.13 forces $p^\upsilon$ to lie closer to the larger bid $s_k$+1 than $p^\epsilon$. Like the McAfee mechanism, it is possible for the market to clear with an ideally efficient allocation regardless of four possible bid arrangements. The upsilon reduction mechanism can only guarantee an ideally efficient outcome in one arrangement, circled in the second arrangement below.

1. $s_k \leq b_{k+1} \leq b_k \leq s_{k+1}$
2. $s_k \leq \left(b_{k+1} \leq s_{k+1}\right) \leq b_k$
3. $b_{k+1} \leq s_k \leq s_{k+1} \leq b_k$
4. $b_{k+1} \leq s_k \leq b_k \leq s_{k+1}$

The upsilon reduction mechanism also circumvents the Myerson and Satterthwaite impossibility result [70] by achieving IR, IC, and BB while also being asymptotically efficient.

By slightly modifying the definition of an efficient trade, $X_v$ can extend the budget balance requirement from merely ensuring non-negative revenue to guaranteeing positive revenue when at least one trade is cleared, all without sacrificing efficiency.

The upsilon reduction mechanism is also the first mechanism that is able to clear the market with ideal efficiency while guaranteeing a positive revenue when at least one trade is cleared.

**Lemma 12** : *The upsilon reduction mechanism, $X_v$, satisfies the property of Individual Rationality.*

*Proof.* Consider the condition $p^v, p^e \in [s_k, b_k]$, all efficient buyers trade at $p_b = p^v$ and all efficient sellers trade at $p_s = p^e$. By equations 4.11 and 4.12 the prices lay strictly $b_{k+1} > (p^e, p^e) > s_{k+1}$. Because of equations 2.10 ($b_1 \geq b_2 \geq ... \geq b_n$) and 2.11 ($s_1 \leq s_1 \leq ... \leq s_m$) buyers will always pay less than their bid while sellers always receive more $b_k > (p^e, p^v) > s_k$ satisfying IR on both sides of the market.

Considering all remaining conditions, traders $s_k$ and $b_k$ are removed from trading. Equation 2.10 states $b_{k-1}$ and all subsequent trading buyers are more competitive than the least favorable buyer: $(b_1, ..., b_{k-1}) \geq b_k = p_b$. Similarly, equation 2.11 states $s_{k-1}$ and all subsequent sellers are more competitive than the least favorable seller $(s_1, ..., s_{k-1}) \leq s_k = p_s$ hereby satisfying IR in both cases. □

**Lemma 13** : *The upsilon reduction mechanism, $X_v$, satisfies the property of Budget Balance.*

*Proof.* Under the condition $p^e, p^v \in [s_k, b_k]$ buyers trade at $p_b = p^v$ and sellers trade at $p_s = p^e$. Equation 4.14 ensures that the prices buyers pay exceed what sellers receive satisfying budget balance under this condition. Under all other conditions the efficient traders $s_k$ and $b_k$ are removed from trading and used as prices for their markets, $b_k = p_b$ and $s_k = p_s$. As the reduced traders are efficient the buyer bid is greater than or equal to the seller bid: $b_k \geq s_k$. Equation 2.12 ensures that for efficient traders the buyer bid exceeds or is equal to the seller bid thus ensuring non-negative revenue, $\Pi(X_v = p_b - p_s \geq 0$. □

**Lemma 14** : *The upsilon reduction mechanism, $X_v$, satisfies the property of Incentive Compatibility.*

*Proof.* All losing traders in the upsilon reduction mechanism, $X_v$, do not pay or receive anything satisfying Definition 37 for a normalized mechanism. Given a trader is in the final allocation, a trader with a more competitive bid will also be in the final allocation. According to Definition 38 this allocation rule is bid monotonic.

Under the condition $p^e, p^v \in [s_k, b_k]$, all buyers above their critical value of $c_i = p^v$ trade and pay their critical value. All sellers below their critical value of $c_j = p^e$ trade and receive their critical value as payment. Under all other conditions the least favorable traders (LFT) are reduced resulting in the critical value for the buyers side being $c_i = b_k$. The critical value for the sellers side $c_j = s_k$. All buyers above (sellers below) this critical value trade and pay their critical value. By theorem 1, $X_v$ is IC. □

**Theorem 7** : *The upsilon reduction mechanism, $X_v$, is prior-free, deterministic, and satisfies the properties of IR, BB, and IC while achieving asymptotic efficiency.*

*Proof.* By Definition 43 the upsilon reduction mechanism, $X_v$, is deterministic because it does not insert randomization into any part of the mechanism. The rules of the mechanism are not dependent on the prior distributions of the buyer's valuation or seller's reservation prices and is able to operate "prior-free" satisfying Definition 42. Lemma 12 proves the mechanism is IR, Lemma 13 proves the mechanism is BB, and Lemma 14 proves the mechanism is IC. At most one trade is reduced from the efficient allocation causing $X_v$ to converge at a rate $1 - \frac{1}{k}$ proving the mechanism is asymptotically efficient. □

So long as efficient trades include those that do not produce any gains ($b_k = s_k$), a positive revenue cannot be guaranteed. Changing Definition 20 to only consider an efficient trade to be one that produce a positive gain from trade, $b_k > s_k$, comes at no cost to efficiency and would allow the upsilon reduction mechanism to guarantee a positive revenue if at least one trade is cleared. This answers teh third and final research questions:

3. Can an IR, IC, and BB trade reduction mechanism be designed to guarantee a non-negative revenue while also achieving asymptotic efficiency?

The upsilon reduction mechanism offers an alternative to the basic trade reduction by using two prices, one for buyers and one for sellers, to generate revenue.

### 4.3.3 *Efficiency and Revenue Performance*

The upsilon reduction mechanism is able to mirror the performance of existing mechanisms. The use of separate prices for buyers and sellers allows the mechanism to generate revenue regardless of whether a trade reduction is necessary.

Three version of the upsilon reduction mechanisms are selected to compare against the McAfee mechanism, basic trade reduction mechanism, and the two SBBA mechanisms.

1. $\epsilon = 0.55, \upsilon = 0.45$ to mimic the McAfee mechanism
2. $\epsilon = 0.1, \upsilon = 0.05$ to mimic the first SBBA mechanism
3. $\epsilon = 0.95, \upsilon = 0.9$ to mimic the second SBBA mechanism

The mechanisms that mimic the SBBA mechanisms have a price range towards the extreme of the possible price selection space, $[s_{k+1}, b_{k+1}]$. These mechanisms also have a smaller price range (0.05 on the normalized scale) than the one meant to mimic McAfee (0.1 on the normalized scale).

The McAfee mechanism acts as an overall baseline for both efficiency and revenue performance in both even and saturated markets. The basic trade reduction shows the upper bound for revenue performance, and each SBBA mechanism serves as an upper bound for efficiency performance in saturated markets.

Figure 4.10 displays the efficiency and revenue performance of three versions of the upsilon reduction mechanism in an evenly distributed market (50% buyers - 50% sellers).

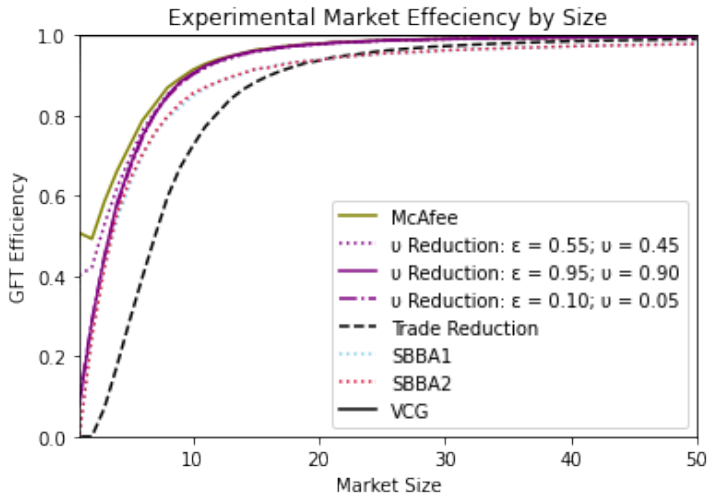
In the single hyperparameter generalized McAfee mechanism, there is only a single price point that efficient traders must accept to clear the market with ideal efficiency. However, with the introduction of a second hyperparameters, a range of acceptable values is created.

The determination of an ideally efficient allocation now depends not on a single point, but by a range. The least efficient buyer must fall above the upper boundary of the range, while the least efficient seller below. The larger the range, the less likely an outcome will result in the ideally efficient allocation, and a trade reduction occur.

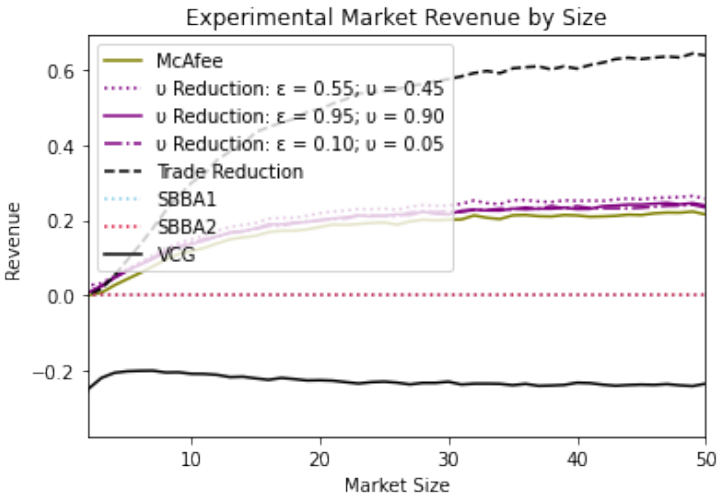
The larger the range, the more revenue the mechanism is able to produce if the market clears with ideally efficiency. Because this larger range makes it more difficult for both prices to be acceptable to the least favorable traders, it also results in a loss of efficiency compared to mechanisms with a smaller difference.

The entire range must fall in between the least favorable traders to clear the market with ideal efficiency: $b_k \geq (p^\epsilon \geq p^\upsilon) \geq s_k$.

Figure 4.10a shows the mechanism with the largest range, $\epsilon = 0.55, \upsilon = 0.45$, achieves less efficiency compared to both of the other upsilon reduction mechanisms.




(a) Efficiency by market size




(b) Revenue by market size

Figure 4.10: Mechanism performance: 50% buyer

A larger range does, however, equate to a larger revenue. Figure 4.10b shows that the mechanism with the largest range ($\epsilon = 0.55, \upsilon = 0.45$) achieves slightly higher revenue then the other upsilon reduction mechanisms.

Depending on the traders prior distributions and market saturation, both the placement and the range dictate the mechanism's relative performance.

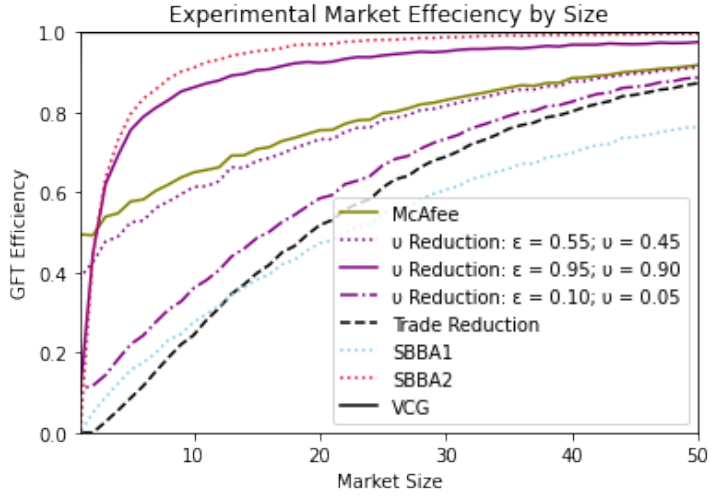
The efficiency and revenue performance from $X_{Mc}$ and $X_{\upsilon}$ are similar when averaged out across $10,000$ runs as shown in Figure 4.10b. In the largest market with size of 50, $X_{Mc}$ produces an average revenue of 0.216 while $X_{\upsilon}$ with $\epsilon = 0.55$ and $\upsilon = 0.45$ produces an average revenue of 0.258.

However, the two mechanisms operate differently in a very important way: The McAfee mechanism, $X_{Mc}$, does not provide a revenue guarantee in each and every market.

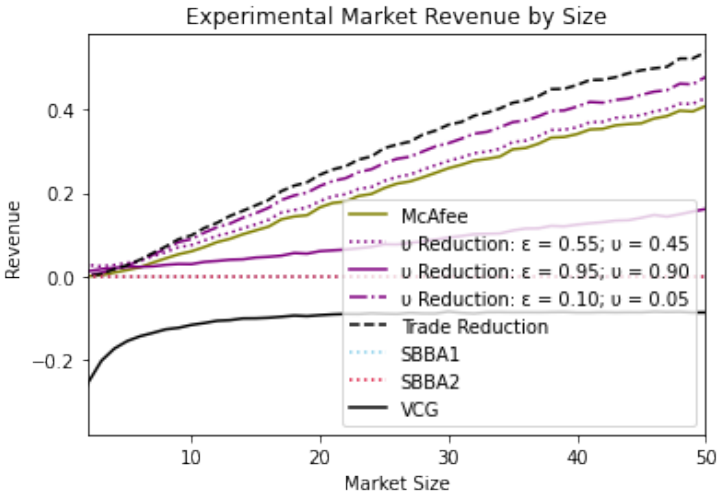
When the McAfee mechanism produces an ideally efficient allocation it does not produce any revenue. Revenue is only generated when a trade reduction is performed. In markets with 50 traders $X_{Mc}$ produces a revenue in $\frac{5036}{10{,}000}$ runs. Roughly 50 percent of markets will not produce a revenue.

The upsilon reduction mechanism, $X_{\upsilon}$, generates a positive revenue in every market except for those in which the least efficient traders are equal $b_k = s_k$. Given bids were drawn from a continuous distribution this did not occur in any of the 10,000 runs.

Figures 4.11 and 4.12 show the same mechanisms in buyer and seller saturated markets. The upsilon reduction mechanism's ability to select both the location and range allows it to mimic the performance of the SBBA mechanism. With one exception, this mechanism is also able guaranteeing a positive revenue when at least one trade is cleared. The exception is when the least favorable buyer and seller bids are equal ($b_k = s_k$).




(a) Efficiency by market size




(b) Revenue by market size

Figure 4.11: Mechanism performance: 90% buyer

In a buyer saturated market, each of the three upsilon mechanism mimic the efficiency performance of their counter parts in Figure 4.11a. The second SBBA mechanism, $X_{SBBA2}$, is extremely efficient in buyer saturated markets due to their price selection, $p_2 = \max(s_k, b_{k+1})$. The selection of a relatively larger price, compared to McAfee, and the SBBA-1 mechanism, results in the ideally efficient allocation occurring much more often. The selection of a single price makes this the most efficient mechanism. The single price however produces a revenue of zero.

Using hyperparameters $\epsilon = 0.95, \upsilon = 0.9$, the upsilon reduction mechanism selects a relatively large pair of prices resulting in similar performance in the efficiency space shown in Figure 4.11a. The use of separate prices on a smaller range (0.05 on the normalized scale) allows this mechanism to produce a positive revenue, that grows with the size market shown in Figure 4.11b.

Similarly, the upsilon reduction mechanism ($X_\upsilon$) with hyperparameters $\epsilon = 0.55, \upsilon = 0.45$ mimics the efficiency performance of the McAfee mechanism in Figure 4.11a. The McAfee mechanism (equivalent to $X_\upsilon$ with hyperparameters $\epsilon = \upsilon = 0.5$) has a slightly higher efficiency due to it's use of a single price. Figure 4.11b shows the slightly higher revenue due to the separate prices from the upsilon reduction mechanism with hyperparameters $\epsilon = 0.55, \upsilon = 0.45$.

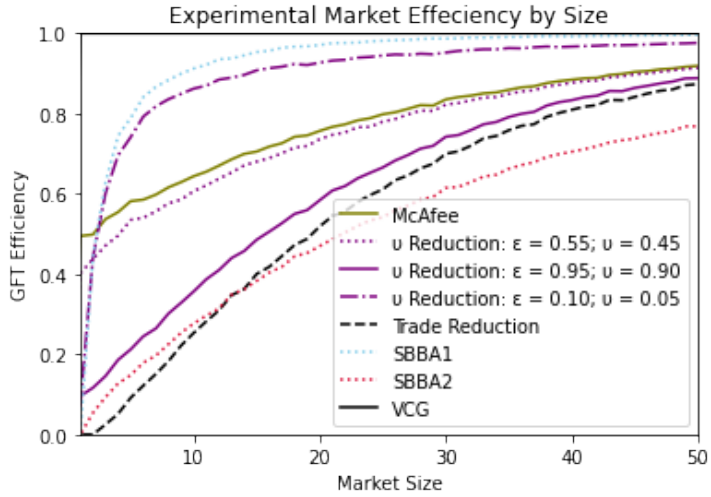
As discussed in Section 3.4 the first SBBA mechanism, $X_{SBBA1}$, performs relatively inefficiently in buyer saturated markets due to its lower price selection, $p_1 = \min(s_{k+1}, b_k)$. Despite the upsilon reduction mechanism with hyperparameters $\epsilon = 0.1, \upsilon = 0.05$ using separate prices for buyers and sellers, this mechanism is able to perform more efficiently then the SBBA-1 mechanism, shown in Figure 4.11a. This occurs because the range of prices are larger than the single price of the SBBA-1 mechanism, which results in the ideally efficient allocation occurring more frequently. Additionally as the market grows, the SBBA-1 mechanism performs less efficiently then the immediate trade reduction mechanism due to it's random removal. Both the upsilon reduction and the basic trade reduction reduce the least favorable traders.

$X_\upsilon$ with hyperparameters $\epsilon = 0.1, \upsilon = 0.05$ achieves a positive revenue that grows as the market grows, while the SBBA-1 mechanism guarantees a revenue of exactly zero from the strong budget balance (SBB) property.

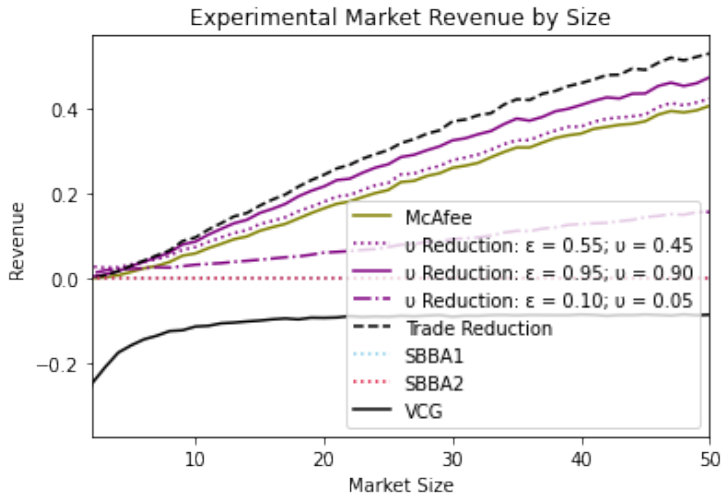
Similar results are shown in the seller saturated market in Figure 4.12.

The upsilon reduction mechanism fills a gap with the ability to mimic the efficiency performance of the SBBA mechanisms in saturated markets while adding on the ability to generate revenue.

The mechanism designer can **optimize** the hyperparameters $\epsilon, \upsilon$ according to the objectives of the mechanism.




(a) Efficiency by market size



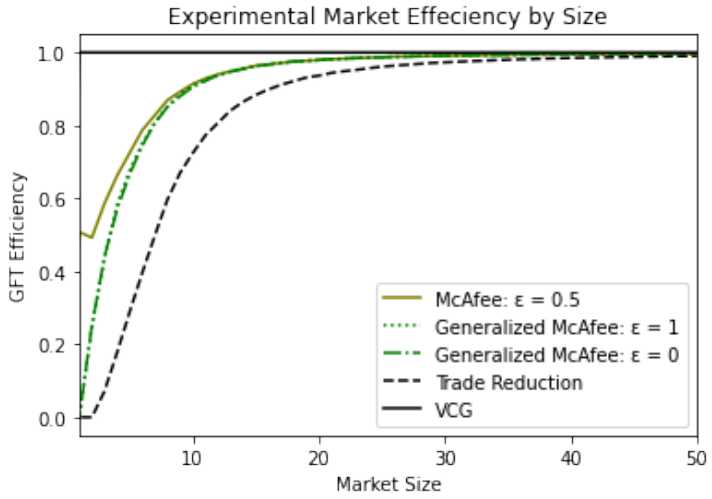

(b) Revenue by market size

Figure 4.12: Mechanism performance: 10% buyer

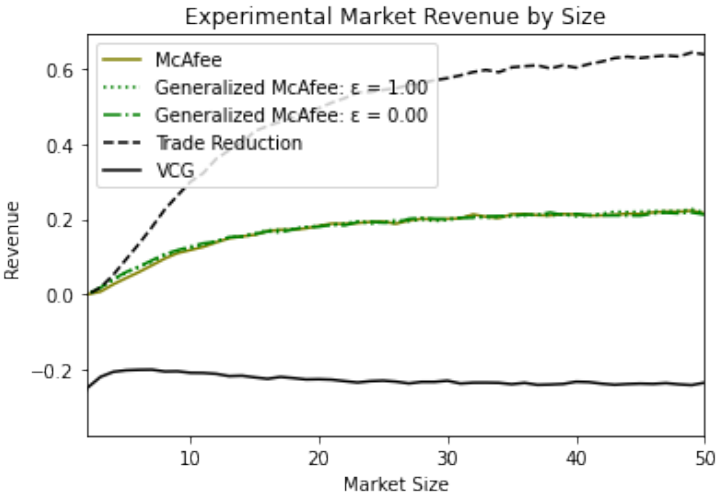
In example, an organization meant to operate as a non-profit can set a minimal revenue to cover the operating costs of setting up the market and optimize the efficiency subject to the revenue constraint. This optimization based on hyperparameters is a novel application of trade reduction based mechanisms. Unlike the previous mechanism introduced by Babaioff and Nissan [7] that requires the knowledge of prior distributions along which buyer and seller private valuation and reservation prices fall; $X_\upsilon$ does not require the use of prior information to operate but allows for it should that prior information be available. $X_\upsilon$ further allows for optimization with limited market knowledge. While prior distributional information may not be available, the expected ratio of buyers may be fully, or partially known.

The $X_\upsilon$ mechanism allows a mechanism designer to achieve asymptotic efficiency, sacrificing minimal efficiency growth in favor of significantly improved revenue and allowing a much more customized set of mechanisms.

The smaller the range between the buyer's and seller's price the more likely the range will result in the ideally efficient allocation. Reducing the range to a single point such that epsilon is equal to upsilon, $\epsilon = v$ would **maximize efficiency**. This also reduces the two hyperparameter upsilon reduction mechanism down to the generalized McAfee mechanism with a single hyperparameter.




(a) Efficiency by market size



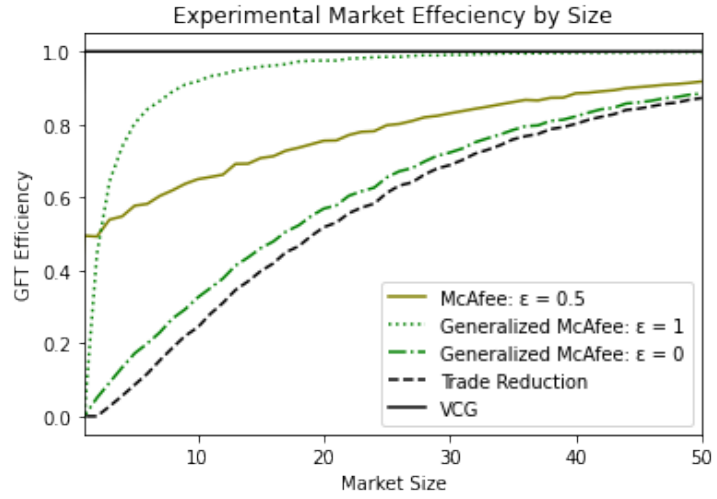

(b) Revenue by market size

Figure 4.13: Generalized McAfee: 50% buyer

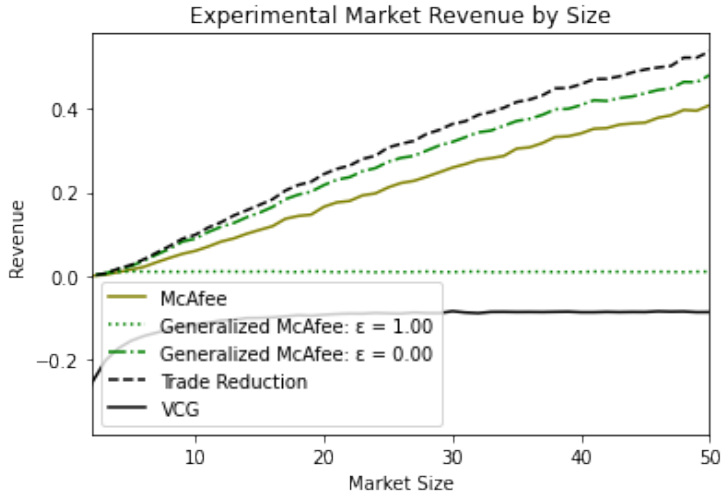
The performance of this unique reduction can also be examined. The Efficiency and Revenue performance of the generalized McAfee mechanism are displayed in Figures 4.13a and 4.13b, respectively. For the hyperparameter $\epsilon$ the extreme values are used where $\epsilon = 1$ and $\epsilon = 0$ to visualize the bounds of the continuous space. Using a uniform distribution for both buyer and sellers private valuation and reservation prices an even split of the market between buyers and sellers in Figure 4.13 shows an extremely small gain in flexibility. As expected in an evenly split market $X_{gen-Mc}$ performs equally whether $\epsilon = 1$ or $\epsilon = 0$ due to the symmetric nature of the distributions and the mechanism.

Figure 4.13a shows the optimal choice of $\epsilon$ to **maximize efficiency** in this market is $\epsilon = 0.5$.

The real benefit of this flexibility is reflected in saturated markets. Consider a buyer saturated market as in Figure 4.14 where the market is made up of 90% buyers.



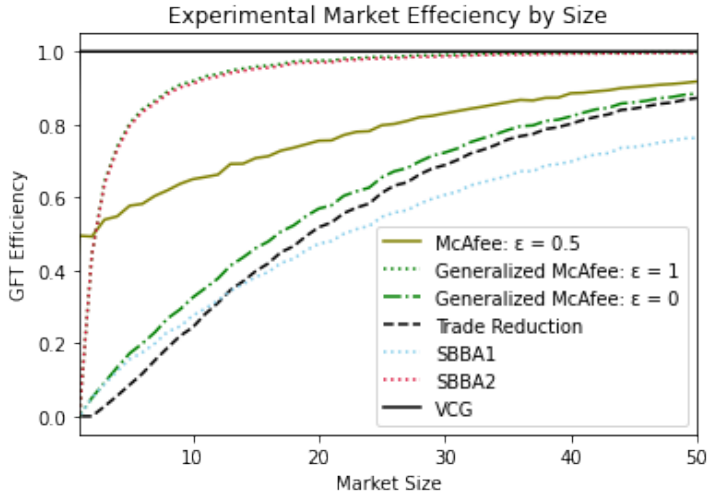

(a) Efficiency by market size



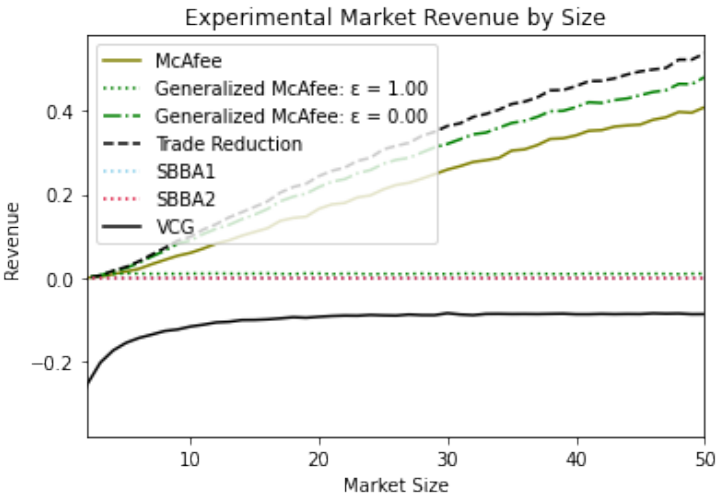

(b) Revenue by market size

Figure 4.14: Generalized McAfee: 90% buyer

Of all the possible values of $\epsilon \in [0, 1]$ the boundaries are set using the extremes where $\epsilon = 1$ and $\epsilon = 0$. The generalization of McAfee, $X_{gen-Mc}$ covers a significant portion of the efficiency space in Figure 4.14a between revenue producing trade reduction mechanism and the efficient VCG mechanism. A significant contribution comes with significantly increased efficiency in a buyer saturated market when $\epsilon = 1$ resulting in $p_\epsilon = b_{k+1}$. Performance resembles that of the second SBBA mechanism with price $p_2 = \max(b_{k+1}, s_k)$ as either $b_{k+1}$ is able to serve as a proper price acceptable to both sides of the market, or a trade reduction is required. Overlaying the SBBA mechanisms onto Figure 4.14 produces the graphs seen in Figure 4.15.




(a) Efficiency by market size




(b) Revenue by market size

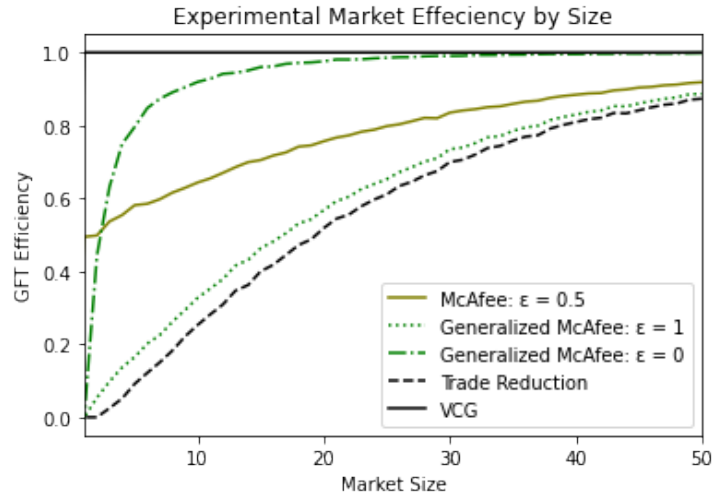
Figure 4.15: Gen-Mc & SBBA mechanisms: 90% buyer saturated

Setting $\epsilon = 0$ results in $p_\epsilon = s_{k+1}$ for the generalized McAfee mechanism. This price hinges on the existence of the inefficient seller $s_{k+1}$ which resembles the price set by the first SBBA mechanism, $p_1 = \min(s_{k+1}, b_k)$. With a buyer saturated markets, the seller side ends up being completely utilized.In other words all available sellers are efficient forcing the mechanism into a trade reduction as $s_{k+1}$ does not exist. Moreover, while the percentage of runs without an inefficient seller $s_{k=1}$ reduces as the market size grows, this is mitigated by the property of asymptotic efficiency. When the market needs it most it does not have it. Without knowledge of the prior distributions this can become an extremely risky undertaking.

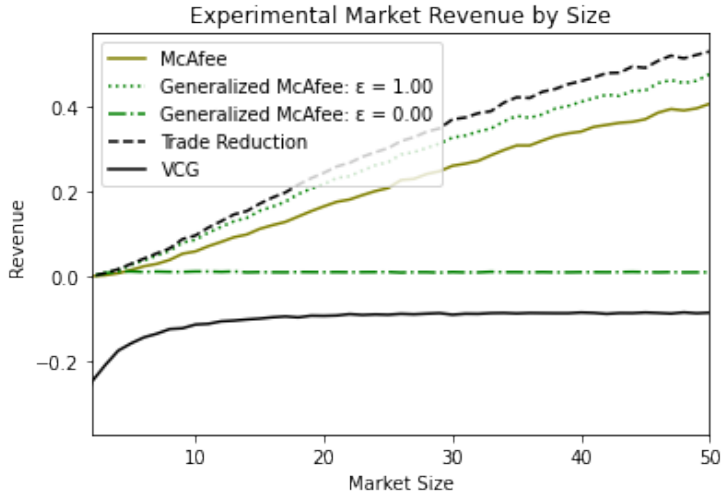
The second significant contribution from $X_{gen-Mc}$ comes in comparison with the second SBBA mechanism. $X_{gen-Mc}$ with $\epsilon = 0$ achieves higher efficiency than both the immediate trade reduction and the first SBBA mechanism as a random reduction is not necessary.

Figure 4.14b and Figure 4.15b shows the natural trade-off between efficiency and revenue. The generalized McAfee mechanism can only produces a positive revenue only in the event of a trade reduction. While difficult to see graphically the $X_{gen-Mc}$ does not achieve SBB as a trade reduction could result in a positive revenue.

For consistency a seller saturated market as seen in Figure 4.16. All the arguments of a buyer saturated market apply analogously to the seller saturated market.




(a) Efficiency by market size




(b) Revenue by market size

Figure 4.16: Generalized McAfee: 10% buyer

An a seller saturated market, the generalized McAfee with $\epsilon = 0$ results in a price much more favorable from an efficiency standpoint, but looses almost all revenue. This mechanism naturally performs almost identical to the generalized McAfee mechanism with $\epsilon = 1$ in a buyer saturated market.

## 4.4 PRE-FILTER WITH MULTI-HYPERPARAMETER MECHANISM

This chapter concludes by integrating both extensions into a generalized trade reduction mechanism. The pre-filter introduced in the first extension addresses the efficiency loss in saturated markets caused by the absence of inefficient traders. The upsilon reduction mechanism, from the second extension, uses hyperparameters to generate revenue both when the mechanism achieves ideal efficiency and when a trade reduction is necessary. The generalized trade reduction mechanism leverages both extensions: it first applies a hyperparameterized pre-filter if all traders on one side of the market are efficient, ensuring no inefficient trader exists. If the pre-filter is not needed, the mechanism defaults to the upsilon reduction mechanism and clears the market accordingly.

### 4.4.1 *Pre-filter Redesign*

The SBBA-1 mechanism, $X_{SBBA1}$, uses the larger, unfavorable for the buyers side of the market, price: $p_1 = \min(s_{k+1}, b_k)$. The mechanism is highly efficient in seller saturated markets due to the high likelihood that an inefficient seller, $s_{k+1}$, exists. However, the mechanism is inefficient in buyer saturated markets, where the absence of $s_{k+1}$ leads to an immediate trade reduction and the loss of a random efficient buyer, further disadvantaging the buyers.

Applying a pre-filter, shown in Section 4.2, involves checking whether all sellers are efficient ($k = m$) and implementing an alternate price estimate, $b_{k+1}$, to try and clear the market with ideal efficiency. For similar reasons, the SBBA-2 mechanism, $X_{SBBA2}$, favors the buyers side of the market. Applying a pre-filter to this mechanisms involves observing whether an inefficient buyer, $b_{k+1}$, exists to clear the market.

Both pre-filters use a single price resulting in a revenue of exactly zero. This is by design as the mechanism aims to maintain the property of *strong* budget balance. The pre-filter can be modified using hyperparameters similar to those used in the upsilon reduction mechanism presented in Section 4.3, the second extension to trade reduction based mechanisms. This allows for different prices to be used for buyers and sellers to generate revenue. Equations 4.16b and 4.16a transform the pre-filter applied to the SBBA-1 mechanism.

$$p_b^m = v^m b_{k+1} + (1 - v^m) * s_{k+1} \tag{4.16a}$$

$$p_s^m = b_{k+1} \tag{4.16b}$$

A superscript $m$ to designate the applied pre-filter prices when all sellers are efficient. The hyperparameter upsilon, $v^m \in [0, 1)$, is used to determine the buyers price in equation 4.16a. Sellers receive the lower price $b_{k+1}$.

Similarly the pre-filter can be applied to the SBBA-2 mechanism. Equations 4.17b and 4.17a use the superscript $n$ to denote when all buyers are efficient. Buyers continue to pay $s_{k+1}$ while the epsilon hyperparameter, $\epsilon^n \in (0, 1]$, is used to determine the sellers trading price. On a normalized scale, the fictitious buyer's bid can be removed from equation 4.17b as the value is set to zero, $b_{k+1} = 0$.

$$p_b^n = s_{k+1} \tag{4.17a}$$
$$p_s^n = \epsilon^n b_{k+1} + (1 - \epsilon^n) * s_{k+1} \tag{4.17b}$$

Each of the $X_{SBBA1-filter}$ and $X_{SBBA2-filter}$ mechanisms individually checks only one side of the market for inefficient traders. To apply a pre-filter to the upsilon reduction mechanism, both sides of the market must be examined to detect any inefficient traders. There are four possible relationships between the number of efficient traders and the total number of buyers and sellers.

1. all sellers are efficient: $k = m$
2. all buyers are efficient: $k = n$
3. all buyers and sellers are efficient: $k = m = n$
4. inefficient traders exist in both sides of the market: $k < min(n, m)$

Hyperparameter values are determined based on the specific market conditions the mechanism encounters. Equations 4.16 are tailored for buyer-saturated markets, which are likely to lead to the first case above ($k = m$). In contrast, equations 4.17 are intended for seller-saturated markets, aligning with the second scenario ($k = n$). The third scenario, where all traders are efficient, is another situation where a pre-filter could be effective. In this case, a mechanism designer might opt for different hyperparameter values to achieve ideal market efficiency because of the use of fictitious order statistics, compared to case four when both sides of the market contain inefficient traders ($k < min(n, m)$).

Equations 4.18 use a superscript $mn$ to denote the case when all traders are efficient.

$$p_b^{mn} = v^{mn} b_{k+1} + (1 - v^{mn}) * s_{k+1} \tag{4.18a}$$
$$p_s^{mn} = \epsilon^{mn} b_{k+1} + (1 - \epsilon^{mn}) * s_{k+1} \tag{4.18b}$$

Equation 4.19 applies a constraint to ensure BB.

$$\epsilon^{mn} \geq v^{mn} \tag{4.19}$$

The pre-filter is not applicable when inefficient traders exists on both sides of the market.

### 4.4.2 *Generalized Trade Reduction Mechanism*

The addition of hyperparameters allows a pre-filter to use separate prices for buyers and sellers. This allows the pre-filter to generate a revenue. A hyperparameterized pre-filter can then be added to the upsilon reduction mechanism to stabilize performance according to market saturation. This novel mechanism, titled the generalized trade reduction mechanism ($X_{GTR}$), represents the culmination of all the previously introduced extensions and is shown in equation 4.20.

$$X_{GTR}(A,P) : (\mathbf{b},\mathbf{s}) \rightarrow \begin{cases} (\mathbf{\Gamma}, \{p_b^m, b_{k+1}\}) & k = m \;\&\; (p_b^m, b_{k+1}) \in [s_k, b_k] \\ (\mathbf{\Gamma}, \{s_{k+1}, p_s^n\}) & k = n \;\&\; (s_{k+1}, p_s^n) \in [s_k, b_k] \\ (\mathbf{\Gamma}, \{p_b^{mn}, p_s^{mn}\}) & k = n = m \;\&\; (p_b^{mn}, p_s^{mn}) \in [s_k, b_k] \\ (\mathbf{\Gamma}, \{p^v, p^\epsilon\}) & k < \min\{n, m\} \;\&\; (p^\epsilon, p^v) \in [s_k, b_k] \\ (\mathbf{\Gamma} - \{b_k, s_k\}, \{b_k, s_k\}) & \text{Otherwise} \end{cases} \tag{4.20}$$

The generalized trade reduction mechanism, $X_{GTR}$, uses a different price for buyers ($p_b$) and for sellers ($p_s$) in every outcome. The mechanism first attempts to clear the market with ideal efficiency. If both prices are not acceptable to all efficient traders ($(p_b, p_s) \notin [s_k, b_k]$), it then performs a trade reduction identical to the one introduced by McAfee. The initial price estimates, $p_b, p_s$, are dependent on the existence of an inefficient buyer ($b_{k+1}$) and seller ($s_{k+1}$).

Equations 4.16 defines the prices to be used when all sellers are efficient ($k = m$). A hyperparameter $v^m \in [0, 1)$ is used to ensure the buyer's payment ($p_b^m$) exceeds the seller's receipt ($b_{k+1}$). In the case $k = n$ the mechanism uses $s_{k+1}$ as the buyer's payment while the seller's receipt ($p_s^n$) is determined by $\epsilon^n \in (0, 1]$ according to equation 4.17a.

If all traders are efficient ($k = n = m$) the mechanism determines buyer and seller prices according to equations 4.18. If both sides of the markets have at least one inefficient trader the mechanism is identical to the upsilon reduction mechanism.

Definition 20 states that a trade between a buyer and seller with the same bid valuation is efficient, $b_k = s_k$. However, this trade does not produce any gains from trade, and therefore can not produce any revenue. This definition limits the generalized trade reduction mechanism from guaranteeing a positive revenue in every outcome in which at least one trade is cleared. Changing the definition to require a positive gain from trade such that $b_k = s_k$ would not effect the GFT, a measure of efficiency. This would allow $X_{GTR}$ to guarantee a positive revenue in every market that results in at least one trade.

**Corollary 3** : *$X_{GTR}$ is prior-free, deterministic, and satisfies the properties of IR, IC, and BB while achieving asymptotic efficiency.*

*Proof.* The total number of buyers ($n$) and seller ($m$) are unaffected by traders bid reports. An inefficient trader can only affect the number of efficient trades by strategically manipulating their bid to join the set of efficient traders. However, since their truthful bid (which reflects their private valuation) places them in the inefficient allocation, such manipulation would result in negative utility for the trader. An efficient trader can only reduce the number of efficient trades by removing themselves from the efficient allocation, which cannot increase their utility.

The same trade reduction technique is applied, regardless of which set of hyperparameterized prices is determined based on the existence of $b_{k+1}$ and $s_{k+1}$. This technique is identical to the one used in upsilon reduction mechanism, which is a generalization of each of the four cases. By theorem 7 each case is then prior-free, deterministic, and satisfies the properties of IR, BB, and IC while achieving asymptotic efficiency.

Because traders cannot strategically manipulate their bids to influence which of the four sets of hyperparameterized prices are used in the attempt to clear the market with ideal efficiency, $X_{GTR}$ is also prior-free, deterministic, and satisfies the properties of IR, IC, and BB while achieving asymptotic efficiency. □

The next section will explore the efficiency and revenue performance of the generalized trade reduction mechanism, $X_{GTR}$. By incorporating a pre-filter, $X_{GTR}$ builds resiliency against market saturation, ensuring consistent efficiency even in imbalanced markets. Additionally, its use of multiple hyperparameters allows for fine-tuning the trade-off between efficiency and revenue, offering flexibility to mechanism designers. The use hyperparameters not only allow for optimization of these objectives but are also able to ensures revenue generation in every market that has at least one efficient trade in the final allocation, regardless of whether a trade reduction is performed or the market clears with ideal efficiency.

### 4.4.3 *Efficiency and Revenue Performance*

The generalized trade reduction (shortened to generalized TR) mechanism is compared against the McAfee, SBBA, basic trade reduction, and VCG mechanisms to assess efficiency and revenue performance. As with previous simulations, each market size is simulated 10,000 times.

A seller's fictitious order statistic is selected to serve as an inefficient sellers in case all seller bids are efficient. To be inefficient the fictitious order statistic must be significantly large to exceed a potential buyer's bid. With known priors, the buyer's distributions upper bound can be

selected as the value for the fictitious seller order statistic ($s_{k+1} = 1$). Because buyers' and sellers' private valuations are drawn from a normalized uniform distribution, McAfee's midpoint estimate between $b_{k+1}$ and $s_{k+1}$ works well, even when fictitious order statistics are used.

Without prior information the selection may exceed the buyer's distribution upper bound to ensure the fictitious order statistic is significantly large enough. To simulate this, a fictitious order statistic is set at twice the upper bound. After renormalization, buyer and seller bids are distributed uniformly between 0 and 0.5, making the fictitious seller order statistic equal to one.

With known priors the fictitious buyer statistic is set to the lower bound of the seller's distribution. Even when the prior distribution are unknown a value of zero can be selected for this fictitious order statistic.

Each generalized TR mechanism requires selecting six hyperparameters. Since buyers and sellers can have different prices, this selection involves determining both the location of the prices within the trade space, $[b_{k+1}, s_{k+1}]$, and the length of the range between buyer and seller prices, $(p_b, p_s)$. The length of this range presents a trade-off between efficiency and revenue. Reducing the range between buyer and seller payments to a single price ($p_b = p_s$) increases the likelihood that the price will be accepted by all efficient traders ($p_b, p_s \in [s_k, b_k]$), thereby **maximizing efficiency**. However, a single price point does not generate any revenue. Expanding this range can increase revenue but also reduces the likelihood that the entire range will be acceptable to both sides of the market.

To assess efficiency and revenue performance a universal normalized window size of 0.01 was chosen for each of the four possible cases in the generalized TR mechanism. Table 4.1 shows the hyperparameter selections.

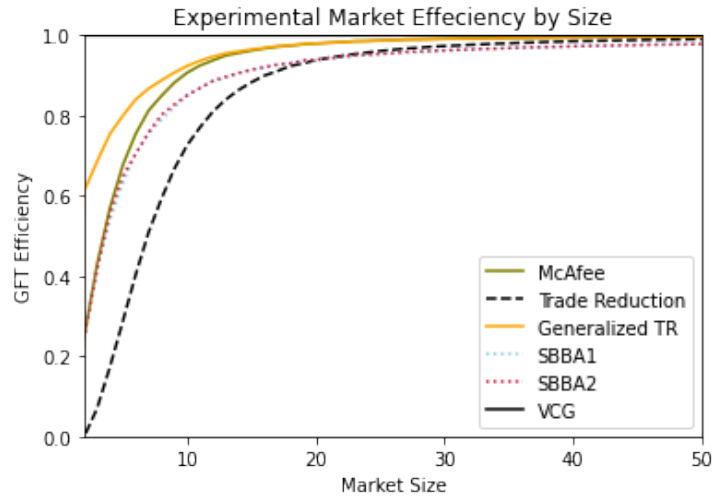
Hyperparamter Selection for $X_{GTR}$

| Case | Value of hyperparameter | Price |
|---|---|---|
| All traders are efficient ($k = n = m$) | $v^{mn} = 0.745$<br>$\epsilon^{mn} = 0.755$ | $p_b^{mn} = 0.255$<br>$p_s^{mn} = 0.245$ |
| All sellers are efficient ($k = m$) | $v^{m} = 0.99$<br>N/A | $p_b^{m} = 0.99 b_{k+1} + 0.01$<br>$p_s^{m} = b_{k+1}$ |
| All buyers are efficient ($k = n$) | N/A<br>$\epsilon^{n} = 0.01$ | $p_b^{n} = s_{k+1}$<br>$p_s^{n} = 0.99 s_{k+1}$ |
| Inefficient traders exist on both sides ($k < \min\{n, m\}$) | $v = 0.495$<br>$\epsilon = 0.505$ | $p^{v} = 0.495 b_{k+1} + 0.505 s_{k+1}$<br>$p^{\epsilon} = 0.505 b_{k+1} + 0.495 s_{k+1}$ |

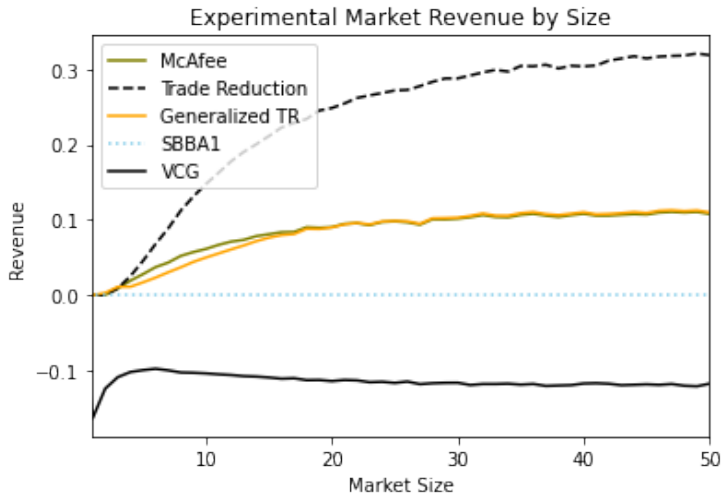
Table 4.1: Hyperparamter selection for $X_{GTR}$

When $k = n$ all buyers are efficient, so the fictitious buyer order statistic is set to $b_{k+1} = 0$. Similarly, when $k = m$ and all sellers are efficient, the fictitious order statistic $s_{k+1} = 1$. The values are substituted into equations 4.18a, 4.18b, 4.16a, and 4.17b shown in Table 4.1.

Figure 4.17a shows the efficiency performance when traders are equally likely to enter the market as buyers and sellers.




(a) Efficiency of GTR mechanism

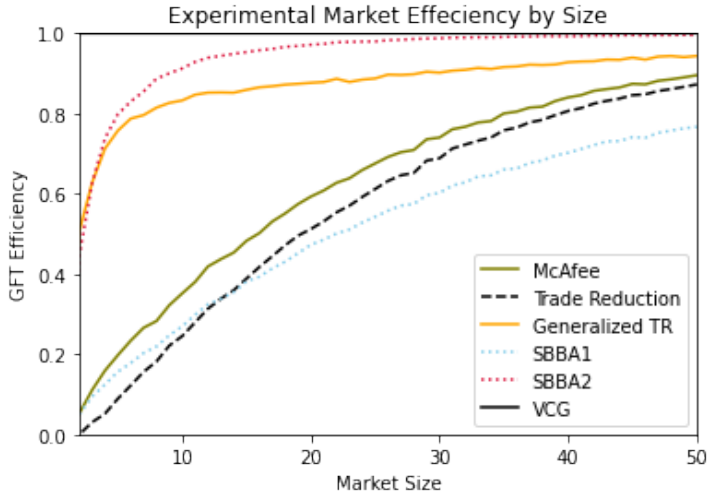



(b) Revenue of GTR mechanism

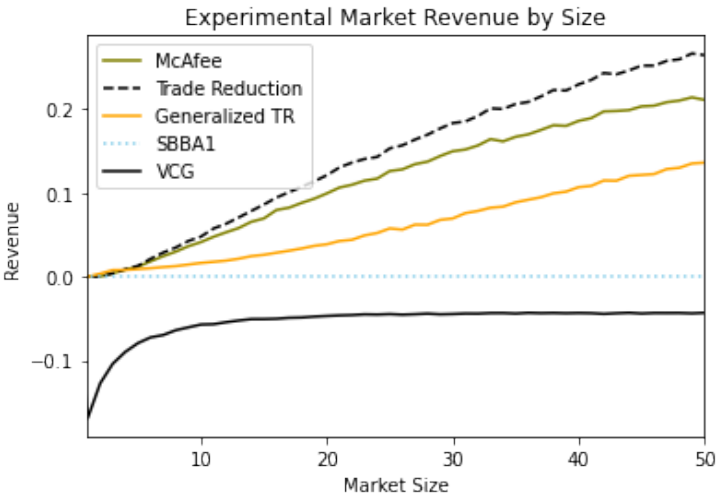
Figure 4.17: GTR performance in even markets

The generalized TR mechanism is able to clear the market with a significantly higher efficiency when the market size is very small because of the added pre-filters. $X_{GTR}$ is the most efficient mechanism in this simulation. This improvement comes at a minimal cost to revenue as seen in Figure in 4.17b.

The revenue generated by both the McAfee and generalized TR mechanisms is quite similar. In smaller markets, there is a slight loss in revenue with $X_{GTR}$; however, as the market grows, $X_{GTR}$ slightly surpasses the revenue produced by the McAfee mechanism. This improvement is due to the generalized TR mechanism's ability to generate revenue even when the allocation is ideally efficient, whereas the McAfee mechanism does not produce revenue in such cases because it uses a single price, $p_0$, to clear the market.




(a) Efficiency of GTR mechanism




(b) Revenue of GTR mechanism

Figure 4.18: GTR performance in saturated markets

Figures 4.18 show the same mechanisms in buyer saturated markets. Traders have a 90% chance of entering the market as a buyer.

Efficiency performance in Figure 4.18a shows the SBBA-2 mechanism as the most efficient, as expected. Although slightly lower, the generalized TR mechanism achieves a similar level of efficiency. Moreover, in early markets, the generalized TR mechanism significantly outperforms the McAfee mechanism [61]. The McAfee mechanism struggles to clear the market with ideal efficiency in smaller markets because it uses a fictitious seller order statistic set at twice the buyer's distribution upper bound [9]. This leads to frequent trade reductions, which is similar to the basic trade reduction.

While the generalized TR mechanism selects hyperparameters to mimic McAfee when inefficient traders exist, the application of prefilters allows $X_{GTR}$ to use different prices when one or both sides of the market are completely efficient. As a result, $X_{GTR}$ shows significantly improved efficiency in early markets and follows a similar efficiency

growth pattern to the McAfee mechanism as the market grows [8]. In larger markets, as inefficient traders become more common, the use of pre-filters decreases, causing the efficiency of the generalized TR mechanism to converge with that of the McAfee mechanism.

This convergence can be more clearly depicted by changing the underlying buyer and seller distribution functions. Figures 4.19 have buyers drawn from a triangular distribution, $F_b$, with minimum value of $\frac{1}{6}$, maximum value of $\frac{1}{2}$ and mode of $\frac{2}{3}$: $F_b \sim Tri(\frac{1}{6}, \frac{1}{2}, \frac{2}{3})$. The fictitious order statistic remains as twice that of the buyer's distribution upper bound, and the market is still 90% buyer saturated. Sellers are distributed $F_s \sim Tri(0, \frac{2}{3}, \frac{1}{6})$.

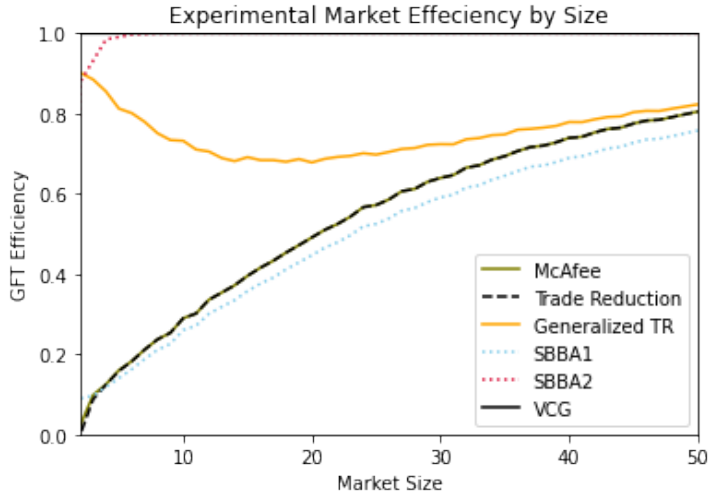



Figure 4.19: GTR efficiency with alternate distributions (90% buyer)

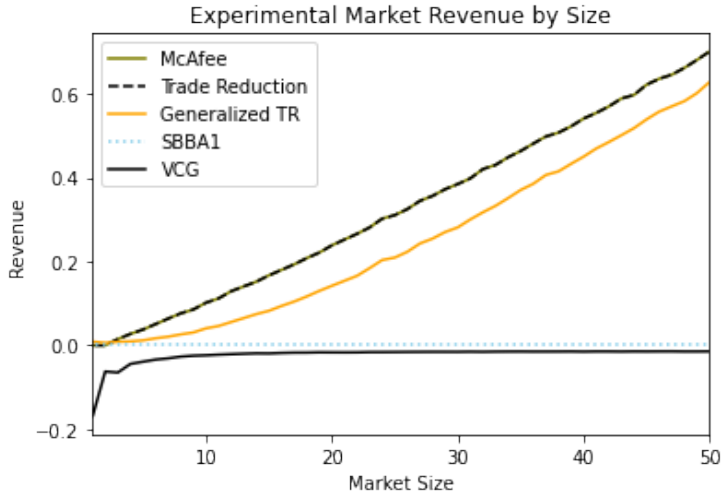



Figure 4.20: GTR revenue with alternate distributions (90% buyer)

According to the alternate buyer and seller distributions, buyers are much more likely to have a private valuation larger than sellers reservation. This means buyers are very interested in acquiring the good/service sellers have to offer.

The buyer saturation further shows this increased demand. This type of market reflects luxury items, in which many buyers and interested in paying premium prices that tend to exceed the cost of making the good/service to be auctioned.

Figure 4.19 shows the SBBA-2 mechanism is nearly ideally efficient, while the McAfee mechanism struggles to clear the market with ideal efficiency, performing similarly to the basic trade reduction mechanism. In this market, the convergence of the generalized TR mechanism is evident. Initially, the pre-filters are highly effective in clearing the market. However, as the market grows and the likelihood of all traders being efficient decreases, the generalized TR mechanism shifts to clearing the market with prices that closely mimic the McAfee mechanism: $p^v = 0.495b_{k+1} + 0.505s_{k+1}$, $p^e = 0.505b_{k+1} + 0.495s_{k+1}$.

The revenue in Figure 4.20 shows similar performance between McAfee, the basic TR mechanism, and the newly introduced generalized TR mechanism.

The generalized TR with parameters according to Table 4.1 is extremely efficient if this simulated luxury market is not saturated with buyers. Figure 4.21 shows the efficiency performance when traders are equally likely to enter the market a buyer as they are a seller.

Figure 4.22 shows $X_{GTR}$ is able to achieve a steadily increasing revenue in addition to the near ideal efficiency. This is in contrast to the near ideal efficiency performance of the SBBA-2 mechanism (which does not produce a revenue) in buyer saturated markets.

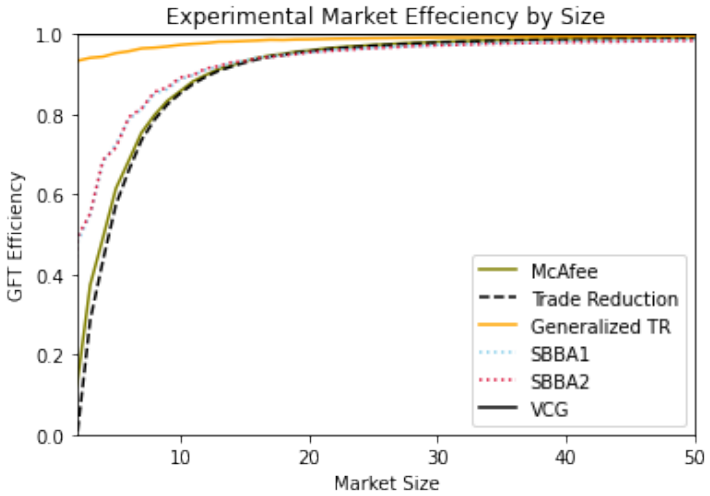



Figure 4.21: GTR efficiency with alternate distributions (50% buyer)

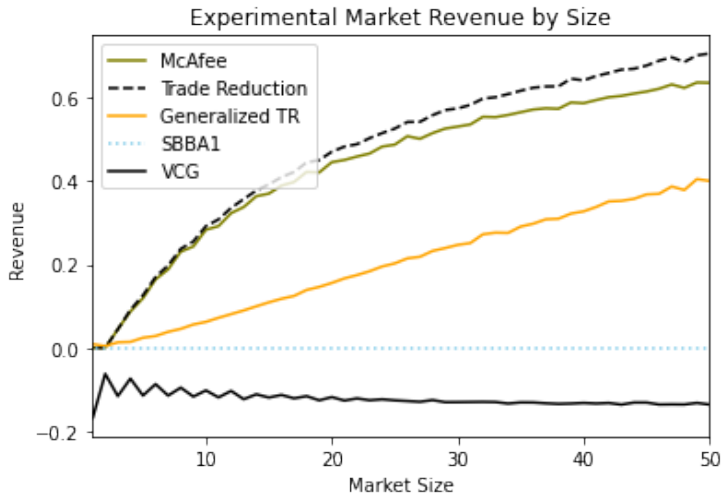



Figure 4.22: GTR revenue with alternate distributions (50% buyer)

The use of a pre-filter builds resiliency into the generalized trade reduction mechanism, $X_{GTR}$ by accounting for the efficiency loss in saturated markets caused by the absence of inefficient traders. $X_{GTR}$ combines the pre-filter with multiple-hyperparameters to generate revenue regardless of whether a trade reduction is performed or the mechanism clears with ideal efficiency. The hyperparameters can be **optimized** to perfectly balance efficiency and revenue generation while maintain the economic properties of IR, IC, and BB.

## 4.5 SUMMARY OF CONTRIBUTIONS

This chapter began by with the introduction of an impossibility theorem that answers the first research question.

1. Is there a normalized, prior-free, and asymptotically efficient trade-reduction mechanism with a bid monotonic allocation rule that achieves both strong budget balance (SBB) and is deterministic?

This theorem defined the parameters that dictate the necessary trade-off between determinism and the economic property of *strong* budget balance.

The final two research questions are concerned with the design of new trade reduction mechanisms.

2. Can a trade reduction mechanism be designed that holds the economic properties of IR, IC, and BB that is resilient to market saturation in terms of efficiency performance?

3. Can an IR, IC, and BB trade reduction mechanism be designed to guarantee a non-negative revenue while also achieving asymptotic efficiency?

These questions are answered with the four novel contributions to the design of mechanisms in the field of operations research. Specifically, the second and third research questions are answered with the second and third research contributions, respectively:

1. The 'Alternate VCG' mechanism, presented in Section 4.1.3, enhances efficiency benchmarks of existing mechanisms while maintaining a near-zero budget. It is resilient to market saturation and generates a slight positive or negative revenue.
2. A "pre-filter", introduced in Section 4.2, that enhances an existing mechanism that detects inefficient traders, allowing the mechanism to improve its resilience to market saturation.
3. The upsilon reduction mechanism ($X_v$), introduced in Section 4.3, can guarantee a revenue in each and every outcome.
4. The generalized trade reduction mechanism ($X_{GTR}$), introduced in Section 4.4, can guarantee a revenue in each and every outcome and detect inefficient traders, allowing the mechanism to improve its resilience to market saturation.

The first contributions presents an alternative to the trade reduction mechanism that is able to circumvent the Myerson and Satterthwaite impossibility result [70]. The 'Alternate VCG' mechanism sacrifices both budget balance and ideal efficiency to enable the **optimization** of both. This mechanism is able to improve upon the efficiency of trade reduction based mechanisms, mimicking the celebrated Vickrey-Clarke-Groves (VCG) mechanism (named after their individual contributions [94], [28], and [42]); however, the mechanism avoids the significant budget deficit, a staple of the classic VCG mechanism.

The "pre-filter" serves as the second research contribution and aims to **maximize efficiency** while ensuring the mechanism is resilient to market saturation. The addition of a "pre-filter" allows the mechanism to maintain the more strict *strong* budget balanced economic property, individual rationality, incentive compatibility, and asymptotic efficiency.

Section 4.3 introduces hyperparameters to both generalize existing mechanisms and guarantee a positive revenue. This goes beyond the definition of the economic property of budget balance, which ensures non-negative revenue. The design of a new mechanism, titled the upsilon reduction mechanism ($X_v$), achieves this without sacrificing efficiency and marks the third contribution of this research.

Finally, the generalized trade reduction mechanism ($X_{GTR}$) combines the pre-filter with the use of hyperparameters into a single mechanism, taking advantage of the contributions of both mechanisms. The introduction of adjustable hyperparameters give greater control over the trade-off space between efficiency and revenue, first characterized by Loertscher et al. [56].

This research explores the trade-off space between revenue and efficiency. By characterizing the space, this research allows a designer to **optimize** a mechanism according the market in which the mechanism is to be implemented. The ability to optimize this trade-off is critical when aiming to balance profit and service, as is the case in the software as a service (SaaS) industry.

This industry is exceptionally relevant in the US military setting that is aiming to maintain ownership of data, architecture, and modular design of software [2]. This allows the government to protect the intellectual property of it's members.

Section 5 discussion the extension of this research to the application of auction mechanism to cloud/edge computing in the SaaS industry, specifically related to military application.

# 5

# EXTENSION: AUCTIONS FOR CLOUD/EDGE COMPUTING IN SOFTWARE AS A SERVICE

The announcement of the 2020 Nobel prize in economic sciences led to a re-invigoration in the field of auction theory with many additional applications sprouting in literature. Emerging technology industries are an ideal application of auction and mechanism design with the ability to determine competitive and mutually satisfactory pricing while achieving highly efficient allocations without market information, obviously lacking in emerging and innovation technology industries. One such emerging technology that is highly active in research is cloud/edge computing. Cloud computing is a model for enabling ubiquitous, convenient, on-demand network access (usually via the Internet) to a shared pool of configurable computing resources [62]. Edge computing is a type of decentralized computing paradigm, which moves data, computation, and storage from the data center to edge nodes of the network [80].

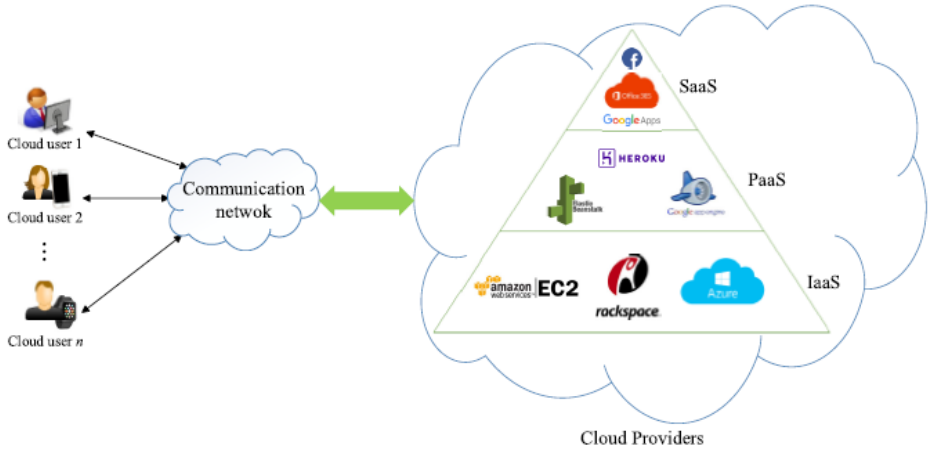



Figure 5.1: Cloud computing service models

Figure 5.1 shows how users interact with cloud service providers [89].

The United States Air Force, as directed by the chief of staff of the air force, has embraced the cloud computing paradigm to manage its data needs [2]. Advances in technology can bring about dramatic changes in military operations and how countries approach warfare, often termed Revolutions in Military Affairs or RMAs [13]. The RMA is a process of discontinuous, disruptive, and revolutionary change, it is nothing less than a fundamental change – a "paradigm shift" – in the character and conduct of warfare. This requires the both the modernization through emerging technologies and the organizational adaptation into operationally relevant environments.

An RMA occurs when the application of new technologies in a significant number of military systems is combined with innovative operational concepts and organizational adaptation in a way that fundamentally alters the character and conduct of a conflict. It does so by producing a dramatic increase in the combat potential and military effectiveness of the armed forces.

Cloud computing can be categorized into five levels of service as seen in Figure 5.2, arranged according to the amount of support offered by the cloud service provider [41].

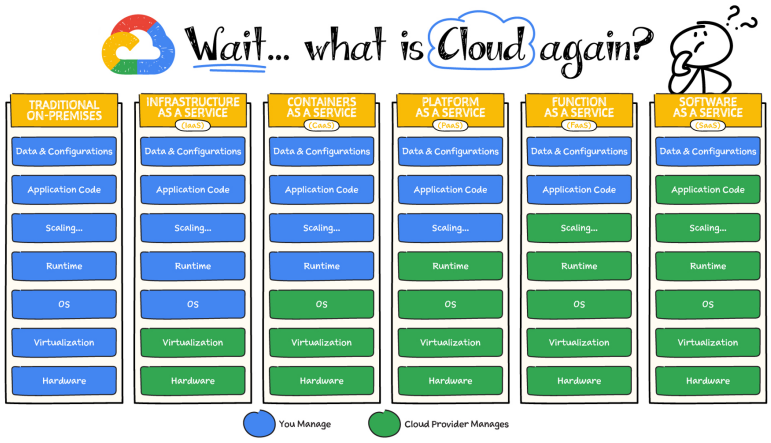



Figure 5.2: Cloud services

Traditional on-premises computing is where an organization manages and maintains its own physical servers and infrastructure. This approach gives businesses full control over their hardware, software, and data security. However, it also means they must handle all aspects of system management, such as upgrades, maintenance, and scaling, which can be costly and resource-intensive. In contrast, cloud services like Infrastructure as a Service (IaaS), Containers as a Service (CaaS), Platform as a Service (PaaS), Function as a Service (FaaS), and Software as a Service (SaaS) offer flexible alternatives, each with varying degrees of control, management, and scalability [77].

1. **Infrastructure as a Service (IaaS)**

   IaaS offers virtualized computing resources over the internet, allowing them to rent servers, storage, and networking hardware instead of investing in physical infrastructure. While users manage their applications and operating systems, the cloud provider handles the underlying hardware, storage, and networking. This model provides flexibility and scalability without the burden of maintaining physical infrastructure [58]. Compared to traditional on-premises setups, IaaS eliminates the need for capital investments in hardware while offering dynamic scaling to meet fluctuating demand.

2. **Containers as a Service (CaaS)**

   CaaS builds on IaaS by providing a platform for managing, running, and deploying containers. Containers package applications and their dependencies into portable units, ensuring they run consistently across different environments [54]. CaaS simplifies container orchestration (e.g., using Kubernetes), which reduces the complexity of managing containers at scale [68]. Unlike traditional on-premises systems, where applications need manual configuration and deployment, CaaS automates these processes and enhances portability and scalability without worrying about infrastructure management.

3. **Platform as a Service (PaaS)**

   PaaS goes one step further by offering a complete platform that includes both the infrastructure and the development tools needed to build, run, and manage applications [99]. Developers can focus on writing code, as PaaS takes care of the infrastructure, operating system, and runtime environment. This significantly reduces the time and effort required to deploy applications. In contrast to traditional on-premises setups, where businesses must manage both hardware and software, PaaS abstracts much of the backend management, allowing faster development cycles and easier scalability.

4. **Function as a Service (FaaS)**

   FaaS, also known as serverless computing, allows developers to run individual functions or pieces of code without managing servers or infrastructure. The cloud provider automatically scales resources up or down based on demand, billing only for the actual compute time used [81]. In contrast to on-premises computing, where servers must be continuously running (even during idle times), FaaS reduces costs by charging only for the exact usage. FaaS is ideal for event-driven applications, where functions are executed in response to specific triggers or events.

5. **Software as a Service (SaaS)**

   SaaS delivers fully functional software applications over the internet. Users access the software through a web browser, and the cloud provider handles everything from infrastructure to data security, application updates, and maintenance [93]. SaaS eliminates the need for businesses to install and manage software locally, as they can rely on the provider for the complete service. Compared to traditional on-premises systems, which require the purchase, installation, and ongoing management of software, SaaS offers a more cost-effective, scalable, and hassle-free solution for a wide range of applications [52].

The US Air Force has begun to completely modernize the way it manages it's data, software, and applications, adhering to the Software as a Service (SaaS) model to reduce the operational burden, allowing it to scale resources and optimize costs more effectively [2].

Edge computing architecture represents a decentralized approach to data processing, where computation and data storage occur closer to the data source, such as Internet of Things (IoT) devices, rather than relying solely on centralized cloud data centers. This architecture reduces latency, enhances real-time data processing, and minimizes the bandwidth needed to send vast amounts of data to the cloud. In edge computing, nodes, or "edge devices," handle computations locally, enabling faster responses for time-sensitive applications like autonomous vehicles, industrial automation, and smart cities.

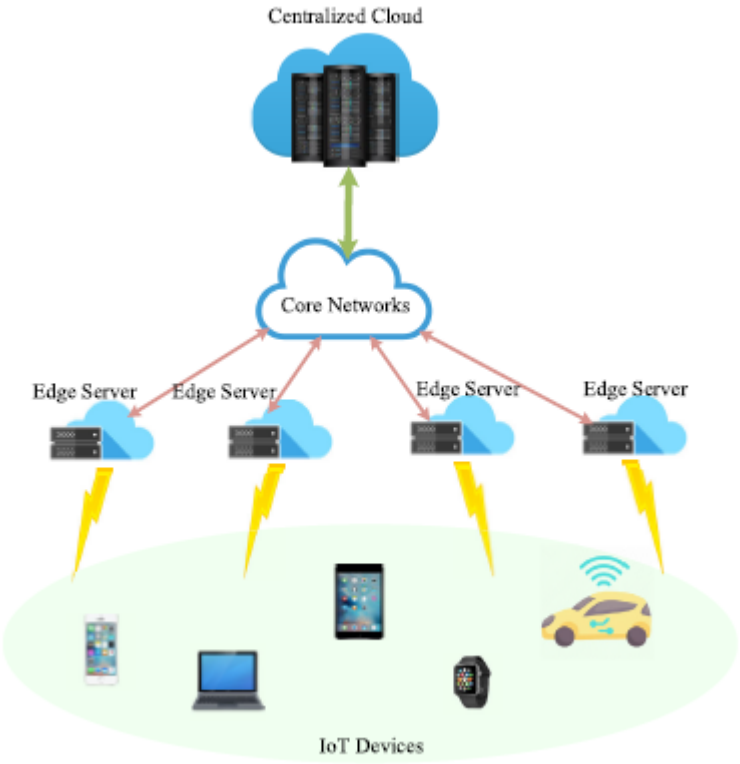



Figure 5.3: Edge computing architecture

Leveraging more distant cloud services improves computational capabilities of Internet of Things (IoT) devices, however, can come at the cost of high latency and mobility-related issues [75]. This can be infeasible for applications requiring fast processing and quick response time such as online gaming, speech recognition, virtual reality, and real-time traffic monitoring. Edge computing, conceptually shown in Figure 5.3, can address these challenges by bringing computing resources to the vicinity of IoT devices [80].

Efficient resource and price management are essential in edge computing systems, especially when integrating IoT devices shown in Figure 5.3.

With millions of IoT devices continuously generating data, ensuring that the limited computational resources at the edge are effectively allocated is critical for maintaining system performance and cost-effectiveness. Due to high maintenance and operational costs, cloud/edge infrastructures require efficient resource management mechanisms for both physical resources (CPU, RAM, storage, bandwidth, etc.) and the provisioning of these resources to users [102].

Price management becomes a significant factor as edge services scale, particularly when balancing between available resources and demand spikes. Mechanisms that optimize pricing based on resource consumption or service priority can ensure that edge computing remains sustainable and accessible for applications requiring real-time processing while avoiding resource bottlenecks that could degrade performance [80]. Efficient allocation and dynamic pricing models enable edge systems to operate cost-effectively while maintaining performance levels needed for IoT-driven operations.

Resource management coupled with pricing present two major challenges in the field of cloud/edge computing. Over-charging or under-charging resources can result in the under-utilization or over-utilization respectively, while inefficient pricing hinders workload distribution over time, required to achieve load balancing. The coupling of these problems is well documented in the field with the problem referred to as market-oriented cloud/edge computing [23].

A successful application of auction mechanism to cloud and edge computing comes in the proposal of DeCloud, a decentralized market framework [103]. This mechanism uses the SBBA mechanism [87] as a foundation as the mechanism has no desire to generate a revenue. A revenue is desirable when it can be used to reward the auctioneer. However, in their setting, "miners" responsible for the algorithm execution are rewarded by cryptotokens emission.

Formulating this using **auction based approaches** yields many advantages seen in Figure 5.4. Individual cloud users act as buyers while providers are the sellers in positions of a cloud service. The price is monetary for this exchange.

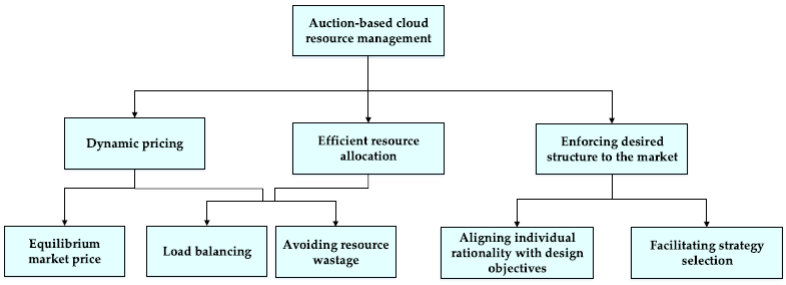



Figure 5.4: Auction Advantages in Cloud Resourcing

Auction-based cloud resource management provides significant advantages for cloud service providers and users alike, primarily through dynamic pricing, efficient resource allocation, and the ability to enforce desired market structures, shown in the second layer in Figure 5.4. By using an auction mechanism to allocate resources, cloud providers can dynamically adjust prices based on real-time supply and demand, creating equilibrium market prices. This approach ensures that resources are priced **optimally**, with the market determining the value of compute power, storage, and bandwidth at any given moment. Dynamic pricing allows for rapid adjustments in response to fluctuations in demand, ensuring that cloud resources are neither over nor under utilized. For instance, during peak usage periods, higher demand can raise prices, encouraging only the most critical workloads to run, while off-peak periods see lower prices, promoting greater resource utilization.

In addition to dynamic pricing, auction-based models allocate resource efficiently, represented with the middle box in the second layer of Figure 5.4. By allowing users to bid for resources, this system ensures that resources are allocated to those who value them most, achieving optimal utilization. This also enables effective load balancing, as resources are distributed based on real-time demand rather than static allocation policies. The dynamic nature of the auction model reduces resource wastage, ensuring that no idle resources sit unused while demand exists elsewhere. This results in an overall increase in cloud infrastructure efficiency, benefiting both the provider—who maximizes resource utilization—and the consumer—who gains access to resources as needed without overpaying for unnecessary capacity.

One of the most critical advantages of auction-based systems is the ability to enforce desired market structures, the final second layer box shown in Figure 5.4. Through carefully designed auction mechanisms, cloud providers can align market dynamics with specific design objectives, such as individual rationality, incentive compatibility, and efficiency. By aligning individual rationality (IR) with the overall goals of the system, users are incentivized to participate and bid truthfully based on their actual needs. The auction system can also guide participants toward certain strategies or pricing behaviors, enabling cloud providers to manage the market's structure effectively. For example, providers may set reserve prices or bidding rules that facilitate strategy selection, guiding participants toward outcomes that are beneficial to both the market as a whole and individual users.

Overall, a major challenge with cloud and edge computing is the efficient resource management and pricing. The execution must be able to take the preferences of cloud/edge providers, users, and the system as a whole into account. With the ability to design according to incentives, auction mechanisms have gained wide attention as a tool for addressing this challenge [89].

Cloud/edge computing requires the consideration of attributes beyond the price of service, such as the quality of service. Cloud/edge computing auctions desire all the following:

- The auction mechanism should be designed to achieve IC to account for strategic behavior and prevent market manipulation [11].
- The auction mechanism should be designed to achieve IR to incentivize users and providers participation [11].
- The auction mechanism should be BB to ensure the market does not run at a deficit. Government settings require the more strict SBB property [89].
- The auction mechanism should be ideally efficient to maximize the social welfare [11].
- The auction mechanism should be multi-attribute in order to consider other important attributes such as Quality of Service metrics beside the pricing of resources [89].
- The auction mechanism should be combinatorial to offer various resources [79].
- The auction mechanism should be online so that it can provide real-time responses to service requirements [90].

The first four of these desires are based on each of the four foundational economic properties described in Section 2.2: Individual Rationality (IR) which guarantees non-negative payoffs to traders; Incentive Compatibility (IC) which incentives traders to truthfully reveal their private information via bids; Ideal efficiency which ensures all trades that result in a non-negative gain from trade (i.e. efficient) are cleared; and finally Budget Balance (BB) which ensures the mechanism produces a non-negative revenue avoid a deficit and incentivizing the existence of the market itself.

The famous Myerson and Satterthwaite impossibility result [70] presented in Section 2.2.7 proves that achieving all four of these economic properties simultaneously is impossible.

Relaxing the desire for ideal efficiency in favor of asymptotic efficiency allows the McAfee mechanism [60] introduced in Section 3.1 to circumvent this impossibility result by sacrificing at most the least favorable trade (LFT). For this reason, the McAfee auction mechanism has provided the golden standard and serves as the foundation in many modern applications beyond cloud and edge computing such as ride sharing [92].

Double auction mechanisms are applicable within cloud and edge computing as the entire system functions as a marketplace where resources are traded. Any marketplace desires an efficient allocation of resources as well as a fair process that attracts both the cloud users and the cloud service providers.

Kumar et al. proposed a Truthful Multi-unit Double Auction (TMDA) mechanism based on the McAfee mechanism [50]. The TMDA mechanism builds on the McAfee mechanism, using it as a foundation to allocate any surplus as a fee for hosting the auction or establishing the cloud market. In cloud markets, auctions are often repeated in quick succession, reducing the size of the auction. This approach serves two purposes: it ensures the market is cleared regularly and generates consistent revenue at regular intervals.

In practice, clearing the market repeatedly increases revenue generation. However, as the market grows, the gap between the least favorable traders narrows. While more trades are cleared, the revenue per trade decreases, eventually reaching a point of diminishing returns. This trend is evident in the market simulations conducted for this research, such as in Figure 4.10b.

Regular market clearing can also lead to imbalanced markets, where one side becomes saturated. This is particularly relevant in cloud and edge computing, which depend on a large user base but have a limited number of resources and service providers.

This setting presents multiple opportunities for improvement based on the contributions from this research. While losing the least efficient trade may have little impact in large markets, regularly clearing the market reduces its size. In smaller markets, this reduction could significantly affect revenue generation, potentially leading to diminishing returns earlier than expected.

The four research contributions presented in order of appearance are:

- Section 4.1.3 presents the 'Alternate VCG' mechanism, which enhances efficiency benchmarks of existing mechanisms while maintaining a near-zero budget. It is resilient to market saturation and generates a slight positive or negative revenue.
- Section 4.2 enhances an existing mechanism by introducing a "pre-filter" that detects inefficient traders and adjusts the mechanism to improve its resilience to market saturation.
- Section 4.3 introduces the design of a new trade-reduction mechanisms that can guarantee a revenue in each and every outcome.
- Section 4.4 introduces the design of a new trade-reduction mechanisms that can guarantee a revenue in each and every outcome and is resilient to market saturation

The first contribution of this research is the introduction of the "alternate VCG" mechanism in Section 4.1. Like the VCG mechanism, this mechanism runs at a budget deficit. This mechanism implements a similar concept to the various trade reduction mechanisms and also sacrifices a minimal efficiency. This additional sacrifice allows the mechanism to significantly reduce the budget deficit to near zero, and even generate a positive revenue in some cases.

The use of this mechanism could reduce the severity of the constraints necessary to extend the mechanism to the combinatorial double auction setting. This mechanism uses a similar concept to the pre-filter and is thus very resilient to market saturation, a situation purposefully created in TCDA mechanism.

Although both the McAfee and basic trade reduction mechanisms achieve asymptotic efficiency, their performance in smaller, saturated markets can differ substantially. The second contribution is the use of pre-filters in the generalized trade reduction mechanism, presented in Section 4.4, greatly enhances efficiency in smaller markets. This is because of the mechanisms ability to implement alternate prices according to the existence of inefficient traders.

The third research contribution is the design of a mechanism that ensures revenue generation every time the market is cleared without sacrificing the possibility of clearing the market with ideal efficiency. In the McAfee mechanism, when the single price estimate results in an ideally efficient allocation, no revenue is generated since buyers and sellers exchange at the same price. To guarantee revenue in every market, a mechanism designer might forgo clearing the market with ideal efficiency and instead immediately reduce the least favorable trade. However, this basic trade reduction approach significantly compromises efficiency, especially in smaller markets.

The use of hyperparameters in the upsilon reduction mechanism, presented in Section 4.3, ensures revenue generation in every instance of the mechanism so long as an efficient trade is cleared. This mechanism is still able to clear the market with ideal efficiency. Although the overall revenue is comparable to the McAfee mechanism, it is generated consistently, while significantly enhancing the efficiency of each market clearing. This consistency and improved efficiency is particularly valuable in cloud and edge computing environments, where maximizing resources and attracting users is crucial.

The fourth and final contribution aligns with the goal of attracting users. When the generalized trade reduction mechanism leads to a trade reduction, the outcome mirrors that of the basic trade reduction: the least favorable traders $(s_k, b_k)$ are removed, and their bids are used to clear trades among the remaining efficient traders.

However, when the generalized trade reduction mechanism achieves an ideally efficient allocation, the trading prices for buyers and sellers $(p_b, p_s)$ are more favorable to the traders, as they fall within the range $[p_s, p_b] \in [s_k, b_k]$. In contrast, during a trade reduction, not only is one fewer efficient trade completed, but buyers end up paying the same or more $(b_k \geq p_b)$, and sellers receive the same or less $(s_k \leq p_s)$, compared to an ideally efficient market clearing.

This comes at a cost to revenue. When the generalized trade reduction mechanism clears all efficient traders, it generates less or equal revenue per trade compared to the basic trade reduction mechanism.

The generalized trade reduction mechanism can only match the revenue of the basic trade reduction if the buyer's price equals the least efficient buyer's bid ($p_b = b_k$) and the seller's price equals the least efficient seller's bid ($p_s = s_k$). This occurs in only one of four possible bidding scenarios: $b_{k+1} \leq s_k \leq b_k \leq s_{k+1}$.

In all other cases, the generalized trade reduction mechanism generates less revenue per trade. Over time, the basic trade reduction mechanism, despite clearing one fewer trade, will surpass the generalized trade reduction mechanism in total revenue. For instance, if the generalized trade reduction mechanism produces 90% of the revenue per trade compared to the basic trade reduction mechanism, the latter will catch up by the tenth trade and surpass the generalized reduction mechanism by the eleventh trade.

The contributions from the design of new mechanisms are graphically depicted in Figure 5.5.

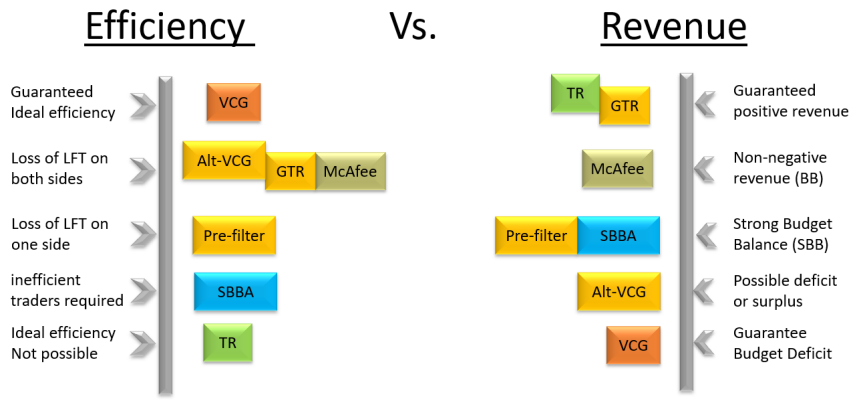



Figure 5.5: Visualization of Contributions

All mechanisms in Figure 5.5 are individually rational (IR) and incentive compatible (IC). This research has focused on providing contributions scoped to the efficiency vs. revenue trade-off space.

The Vickrey-Clarke-Groves (VCG) mechanism [94], [28], [42], colored in orange, is ideally efficient and serves as the upper bound in terms of efficiency. Achieving ideal efficiency comes at a cost to revenue as the VCG mechanism is guaranteed to run at a budget deficit, marking the lower bound for revenue.

The basic trade reduction (TR) mechanism, used in the $\alpha$ reduction mechanism [7], originates from the McAfee mechanism [61]. This TR mechanism, colored in green, immediately reduces the least favorable trade (LFT) making it impossible to clear the market with ideal efficiency. While possible for either of the SBBA mechanisms to achieve lower efficiency due to their random removal of a trader on one side of the market, the TR mechanism acts as the lower bound of efficiency space in Figure 5.5 due to the guaranteed inability to clear the market with ideal efficiency.

The TR mechanism does serve as the upper bound of the revenue space due to the guarantee of a positive revenue so long as the mechanism clears with multiple efficient trades and positive gains from trade.

The disclaimer requiring a positive gain from trade is only necessary when the definition of an efficient trade includes one that does not produce a gain from trade (GFT), i.e. price determining buyer and seller bids are equal. A change in definition to only consider trades that result in a positive GFT, i.e. buyer bids are larger than seller bids, would come at no cost to the measure of efficiency, GFT.

Both of the SBBA mechanisms [87] extend the McAfee mechanism to achieve the economic of strong budget balance (SBB), which guarantees a revenue of exactly zero. This comes at a cost in efficiency as the mechanism introduces the random removal of one side of the market in case of a trade reduction. In terms of the efficiency vs. revenue trade-off space this mechanisms is less efficient and has a lower revenue than the McAfee mechanism.

The first SBBA mechanism is designed to operate extremely efficiently in buyer saturated markets due to it's higher price compared to the second which is designed for seller saturated markets. As a result, both of these mechanisms are especially vulnerable to the market saturation they were not designed for. This leads to an efficiency loss which then compounds due to their use of randomized reductions.

Because these mechanisms carry the SBB property, this also means the loss in efficiency does not come with an increase in revenue. This lack of resiliency provides an opportunity for improvement with the addition of a pre-filter. The use a pre-filter improves the efficiency of the SBBA mechanisms while maintaining the SBB property.

The mechanisms introduced in this research are colored gold in Figure 5.5. While not depicted on the graph, all of the newly designed mechanisms are resilient to market saturation.

As the name suggests, the alternate VCG mechanism (Alt-VCG) is able to achieve the highest efficiency outside of a guarantee of ideal efficiency. This mechanism sacrifices both the economic property of ideal efficiency and budget balance to optimize the performance of both. This mechanism is sacrifices a minimal amount of efficiency and produce a near-zero revenue. The mechanism is able to generate both a positive or negative revenue depending on the market. This mechanism significantly improves upon the guaranteed budget deficit of the VCG mechanism through a small sacrifice in efficiency.

The generalized trade reduction (GTR) mechanism incorporates multiple hyperparameters to create a positive revenue similar to the TR mechanism. While the TR mechanism does produce a higher revenue, the GTR mechanism is able to mimic their performance while achieving a higher efficiency. This is the first mechanism able to both achieve ideal efficiency and produce a positive revenue in the same outcome.

**Designing auctions** specifically for cloud and edge computing presents unique challenges [89]. These extend beyond the assumptions of the mechanisms introduced in Chapter 2 and discussed in Chapter 4. The current mechanisms are based on discrete, bilateral trade settings with a single payment attribute, such as price, as defined in Definitions 15 and 16 [34]. Cloud and edge computing environments may require continuous, online auction mechanisms. Additionally, bilateral trade assumptions are too restrictive, as these settings often involve multiple bundled services rather than a single identical service [19]. Consequently, designing auctions for these environments may need to incorporate **combinatorial approaches** to handle complex service bundles [79].

# 6

# OUTLOOK AND CONCLUSION

The research presented here is falls into the economic field of study known as auction theory, which can be further classified as a subfield within the broader spectrum of non-cooperative game theory. According to Krishna's auction theory textbook "an auction will be understood to include the process of procurement via competitive bidding" [49]. The focus of this research is on the more complex double auction which involves both buyers and sellers simultaneously.

The McAfee mechanism is a cornerstone in mechanism design, circumventing the Myerson and Satterthwaite impossibility result through a trade reduction. This discrete, bilateral trade mechanism ensures individual rationality, incentive compatibility, budget balance, and asymptotic efficiency. The two SBBA mechanisms extend the McAfee mechanism to *strong* budget balance by splitting the VCG payment rule into two mechanisms, which requires a randomized reduction to one side of the market to maintain the economic property of incentive compatibility.

This research introduces several new mechanisms that extend and generalize both the McAfee and SBBA mechanisms. Any application that uses the McAfee mechanism but opts for trade reduction to increase revenue rather than achieving ideal market efficiency could benefit from applying the generalized trade reduction mechanism. This is particularly true if the mechanism designer aims to incentivize market participation by lowering buyer prices and increasing seller revenues compared to the basic trade reduction mechanism.

One vulnerability of the SBBA mechanisms is their performance in buyer- or seller-saturated markets. By splitting the VCG payments, each SBBA mechanism performs highly efficiently in one type of market saturation but struggles in the other. This variation arises from the presence of inefficient traders. In saturated markets, the undersaturated side is often fully consumed by efficient trades. Each SBBA mechanism tries to clear the market with an inefficient trader from one side of the market. The lack of an inefficient trader requires the mechanism to implement a trade reduction and reduces efficiency. Conversely, the presence of an inefficient trader (on the side of the market the mechanism is considering) leads to the mechanism clearing the market with ideal efficiency.

This setting also presents opportunities for improvement based on the research contributions presented in Section 4.2. Both of the SBBA mechanisms are vulnerable to market saturation.

As the design of the SBBA mechanisms is to not generate revenue, the goal becomes **maximizing** efficiency. Selecting the wrong the SBBA mechanism can results in significantly lower efficiency as shown in Figure 4.8 for example. Repeated implementation can also result in any given market being saturated by one side of the market or the other. Applying a pre-filter to the SBBA mechanism significantly improves their resiliency to saturation fluctuations.

This research proposed the use of pre-filters to counter this lack of resiliency to market saturation. The pre-filter observes whether an inefficient trader exists and adjusts the price estimate accordingly. Applying the pre-filter to the SBBA mechanism creates the first mechanisms resilient to market saturation while maintaining the desired economic properties of IR, IC, SBB, and asymptotic efficiency [87]. This adjustments introduces in resiliency and stability to markets [40].

The generalized trade reduction mechanism applies hyperparameters to generalize the McAfee mechanism [60]. The use of hyperparameters allow for distinct price estimates for buyers and sellers. These replace the single price estimate of the McAfee mechanism leading a mechanism that can generate a revenue both when the allocation is ideally efficient and results in a trade reduction. Online auctions are often repeated creating space for learning on some prior Bayesian knowledge agents that may or may not be known. The generalization and introduction of hyper parameters in this research introduces the environment for mechanisms to learn and adjust according to trends [71].

This research contribution goes beyond the budget balance requirement from merely ensuring non-negative revenue to positive revenue without sacrificing efficiency. This extends existing work by both removing randomization and providing a positive revenue guarantee for each and every market that results in efficient trades [7]. The generalized trade reduction mechanism is the first discrete, bilateral trade mechanism that is able to clear the market with ideal efficiency while generating a positive revenue and maintain the economic properties of individual rationality and incentive compatibility.

These new mechanisms build upon the foundations of the McAfee and SBBA mechanisms and can serve as replacements for them. The McAfee mechanism is very well cited and is used a foundation for a plethora of applications. A valuable next steps would be to conduct extensive survey to identifying where mechanisms proposed with this research can replace either the McAfee or SBBA mechanism to improve results.

The application of auction theory to resource allocation in mobile edge networks and cloud computing extend beyond the bilateral trade setting to the combinatorial double auction setting. This setting allows for the extension of the VCG mechanism to be implemented without causing a budget deficit [96].

A combinatorial double auction allows multiple goods/services to be bundled together both for purchase and sale. Certain restrictions can be placed to ensure the market does not produce a negative revenue. The TCDA mechanism for example introduces a linear programming (LP)-based padding method for allocation [57]. Mobile devices only offload tasks to edge servers (ESs) in the proximity with various requirements, and ESs only serve their neighboring MDs with limited resources. This concept aims to intentionally create imbalances between supply availability and demand requirement.

The introduction of hyperparameters in Chapter 4 allows **learning algorithms** to be integrated into mechanisms based on trade reduction concepts [71]. By strategically setting these hyperparameters, it is possible to minimize significant losses in efficiency, thereby enhancing the rate at which these mechanisms approach asymptotic efficiency.

The "alternate VCG" mechanism has potential to improve existing combinatorial double auction mechanisms that rely on the VCG mechanism. The "alternate VCG" mechanisms ability to reduce the budget deficit to near zero, and even generate positive revenue in certain cases could reduce the severity of the constraints necessary to extend the mechanism to the combinatorial double auction setting.

A final particularly interesting avenue for future research would be the extension of the proposed mechanisms from the discrete, bilateral setting to more complex environments, including multi-unit, multi-lateral, and continuous settings. In multi-unit auctions, where multiple units of a good are sold, and multi-lateral markets, where multiple buyers and sellers interact simultaneously, the design and implementation of mechanisms need to account for additional layers of complexity, such as bundling, package bidding, and coordination among participants. Continuous settings introduce further challenges, requiring mechanisms to handle an infinite range of possible bids and prices, necessitating sophisticated mathematical and computational approaches. Extending the mechanisms to these settings could have significant impacts on efficiency and fairness in practical applications, such as cloud computing resource allocation, energy markets, and large-scale procurement processes. By addressing these complexities, researchers can enhance the robustness and applicability of the mechanisms, potentially improving market outcomes and participant satisfaction across a variety of real-world scenarios.

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# A

# ADDITIONAL TRADE REDUCTION MECHANISMS

Additional mechanisms adjacent to this line of research exists in literature that achieve the same required properties of IR, IC, and BB. The first series of mechanisms come from Blumrosen and Dobzinski [14] which was later extended to achieve asymptotic efficiency for bilateral trades [15]. The authors focus on the social welfare rather than the gains from trade (GFT) as a measure of efficiency and aim to quantify an approximation of the optimal social welfare. While similar these objectives can lead to different results. The second [6] adds an additional constraint to the mechanism that does not allow buyers to bid above their valuations or sellers to bid below their private reservation prices. With this constraint to maintain truthfulness their mechanism only needs to prove one direction such that buyers do not have an incentive to bid below their valuation and sellers do not have an incentive to bid above their reservation price to potentially effect the price.

## A.1 NO-LOSS CONSTRAINED DOUBLE AUCTION

The mechanism introduced by Anbarci and Roy is based on the classic McAfee mechanism $X_{Mc}$ and extends it to the strong budget balance (SBB) case using randomization in the payment rules [6]. Their new mechanism referred to as the "Double Auction with No-Loss Constrained Traders", $X_{NLC}$, achieves asymptotic efficiency equal to that of $X_{Mc}$.

The main limitation with this mechanism comes with the requirement of an additional constraint that limits the applicability of the mechanism to markets in which the constraint can be enforced. The constraints is called a No-Loss-Constraint (NLC) and assumes that the traders are completely risk averse to potential losses. Rather than design a mechanism that ensure IR is met, $X_{NLC}$ assumes "stricter restrictions where buyers never bid above their valuations or sellers never ask below their costs" [6]. The mechanism continues to assume the bid order statistics from equations 2.10 and 2.11.

$$b_1 \geq b_2 \geq ... \geq b_n \tag{2.10}$$

$$s_1 \leq s_1 \leq ... \leq s_m \tag{2.11}$$

The efficient number of traders are defined by equations 2.12 and 2.13 according to the allocation rule $k \leq \min\{m, n\}$.

$$b_k \geq s_k \tag{2.12}$$

$$b_{k+1} < s_{k+1} \tag{2.13}$$

The NLC mechanism determines the outcome according to two conditions. The first considers the case when either all buyers, all sellers, or both are efficient traders, $k = \min\{n, m\}$. Under this condition the trade is immediately reduced and a price lottery is implemented.

In the second condition at least one inefficient trader ($b_{k+1}$ and $s_{k+1}$) must exists on each side of the market such that $k < \min\{n, m\}$. The mechanism then checks the McAfee price according to equation 3.3: $p_0 = \frac{1}{2}(b_{k+1} + s_{k+1})$. If $p_0 \in [s_{(k)}, b_{(k)}]$ the market clears with all $k$ efficient traders at price $p_0$. If an efficient trader does not exist the least favorable traders are reduced.

If the McAfee price is not acceptable to both sides of teh market, $p_0 \notin [s_{(k)}, b_{(k)}]$, a trade reduction is performed and the price lottery is implemented to achieve SBB. The formal mechanism is shown in Equation A.1.

$$X_{NLC}(\mathbf{b}, \mathbf{s}) = \begin{cases} (\mathbf{\Gamma} - \{b_k, s_k\}, \{p, p\}) & k = \min\{n, m\} \\ (\mathbf{\Gamma}, \{p_0, p_0\}) & k < \min\{n, m\} \& p_0 \in [s_k, b_k] \\ (\mathbf{\Gamma} - \{b_k, s_k\}, \{p, p\}) & k < \min\{n, m\} \& p_0 \notin [s_k, b_k] \end{cases} \tag{A.1}$$

The main difference between $X_{NLC}$ and $X_{Mc}$ comes with the pricing rule when a trade reduction is necessary. The McAfee mechanism reduces the least favorable traders (LFT), $b_k, s_k$ and uses these bids as the prices for each side of the market. Given that these traders are efficient, the price the buyers pay covers the price the sellers receive, ensuring BB ($p_b = b_k \geq s_k = p_s$).

To achieve SBB $X_{NLC}$ instead implements a price lottery where with probability $q$, all winning traders $\{b_1, ..., b_{k-1}\}, \{s_1, ..., s_{k-1}\}$ clear at price $p = b_k$ and with probability $1 - q$, all winning traders clear at price $p = s_k$.

Implementing a price lottery in this way requires the NLC mechanism to loose the *ex-post* IC economic property. When the market clears using the buys bid as a price, $p = b_k$, such that $q = 1$, then seller $s_k$ is incentived to report a bid lower than their reservation so as to be included in the winning allocation and incur a positive utility.

Similarly, buyer $b_k$ is incentived to report a bid higher than their valuation so as to be included in the winning allocation and incur a positive utility when $q = 0$ and $p = s_k$.

Traders bidding in this manner run the risk of potentially paying a price that results in a negative payoff. This behavior is in violation of the NLC and thus restricted by this mechanism. This inherently assumes that traders are extremely risk averse wanting to avoid negative payouts at all costs.

A noticeable difference between $X_{NLC}$ and $X_{Mc}$ is $X_{NLC}$ does not leverage the fictitious order statistics used in $X_{Mc}$. This results in an expected loss of efficiency when the trade is immediately reduced in $X_{NLC}$, while $X_{Mc}$ is still able to calculate a price $p_0$ that may result in an ideally efficient market clearing. This situation arises more prominently in smaller sized markets, in fact with a mid-size market this loss in efficiency can be unnoticeable. In an even market the likelihood that all 25 buyers and sellers are efficient resulting in the potential improved efficiency by McAfee over the NLC mechanism mentioned here is extremely rare. In the 10,000 simulations this did not occur even once resulting in both mechanism achieving identical efficiency as seen in Figure A.1.

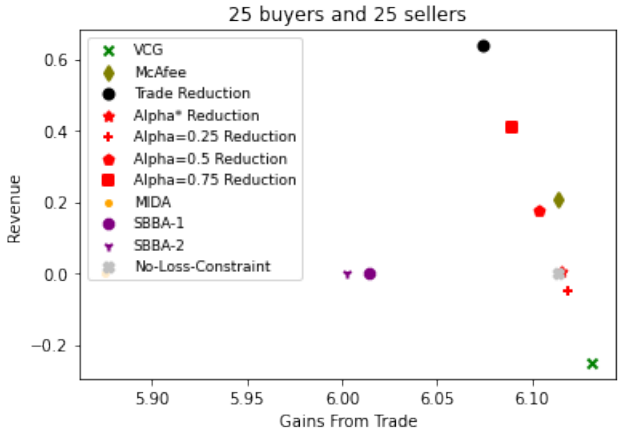
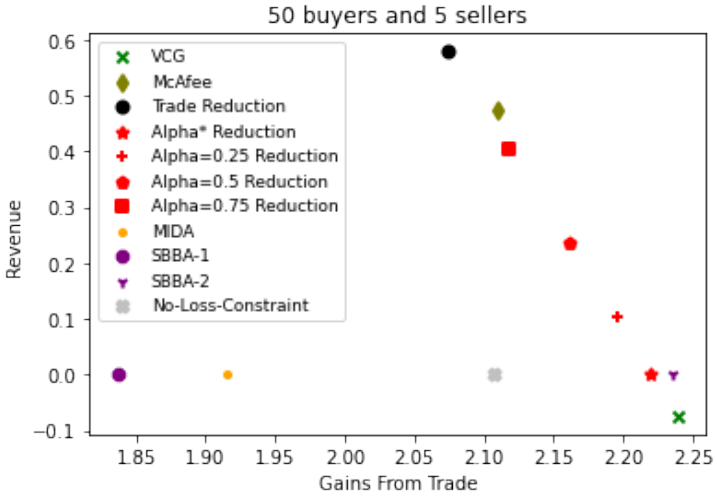
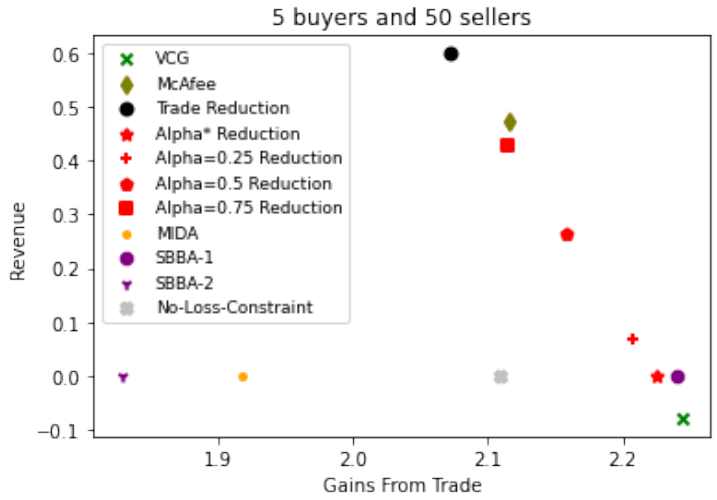



Figure A.1: 25 buyers and 25 sellers

This is also difficult to notice in the saturated markets seen in A.2 because the McAfee mechanism tends to result in a significant amount of trade reductions putting the efficiency achieved only slightly above the trade reduction mechanism. This is because the saturated markets drive the price toward the saturation side resulting in the all trades from the opposite side going through often. Using a single fictitious trader with an inefficient trader results in the McAfee price likely to fall outside of the required bounds for an ideally efficient allocation $p_0 \notin [s_k, b_k]$.




(a) 50 buyers and 5 sellers




(b) 5 buyers and 50 sellers

Figure A.2: GFT performance in saturated markets

In the saturated markets only $\sim$ 3% of runs result in the McAfee mechanism achieving ideal efficiency while the NLC mechanism performs a trade reduction. It is helpful to view the efficiency as the market grows, dropping the comparison with revenue as the NLC mechanism is SBB.

To smooth out the graph increase the number of runs from 1,000 to 10,000 iterations of each market size randomizing whether the trader that enters the market is a buyer or a seller. For example relieve the requirement for there to be exactly 25 buyers and 25 sellers in a market with 50 traders. There is instead a 50% chance that a trader is buyer and 50% chance they are a seller. Continue to draw buyers and sellers from a uniform distribution ranged from 0 to 1.

Figure A.3 shows the efficiency growth up to market size of 50. By relieving the requirement for an exactly even distribution of buyers and sellers, the overall efficiency dropped significantly.

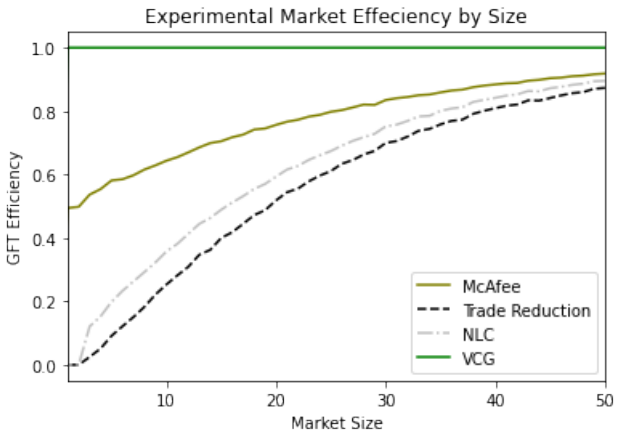



Figure A.3: Efficiency by market size

Table A.1 shows the achieved efficiency percentage relative to the ideal efficiency of VCG.

50 Trader Market (10,000 runs)

| Mechanism | Efficiency Percentage |
|---|---|
| VCG | 100 |
| McAfee | 91.85 |
| NLC | 89.57 |
| TR | 87.34 |

Table A.1: Trader Market

While the efficiency NLC mechanism eventually caught up with that McAfee, Figure A.3 shows a significant difference in the smaller market sizes. This is especially true when the ideally efficient outcome consists of a single trade. The McAfee mechanism has a chance at executing the trade while both the TR and NLC mechanism immediately implement a trade reduction causing a significant hindrance to efficiency.

## A.2 MEDIAN MECHANISM

Two standard definitions of economic efficiency exist. While not necessarily competing, when **optimized** these can lead to different results. The classic measure of economic efficiency is the gains from trade (GFT) which aims to only capture the added utility, or gain, that is produced when a trade is realized [60], [7], [94], [28], and [42]. That is if no trade is executed than there is no change from the market's existence and therefore not considered as a measure of efficiency. An alternate measure more closely tied to computer science is that of social efficiency which aims to place the good/service $g$ in the hands that value it the most. That is seller's that do not sell their $g$ because of a high reservation price contribute to the measure of social welfare. The main reason for this use as a measure of efficiency is for the purpose of approximation; "no DSIC, budget balance and individual rational mechanism can guarantee a constant approximation to the optimal GFT" [15]. In order to approximate efficiency a new measure was required and lead to the social welfare being used.

With approximation as the goal, a series of ex-post IR, IC, and BB mechanisms [14] and [15] use the social welfare as a measure of efficiency. The "main goal is to design markets that allow an efficient reallocation of resources" [14]. The first suggested mechanism is called the *median mechanism* as a single trade price, $p_d$ is set equal to the median of the distribution from which seller's private reservation prices fall. The trading allocation is then determined by finding the number of traders $d$ at which $p_d \in [s_d, b_d]$ and $p_d \notin [s_{d+1}, b_{d+1}]$. The authors continue to show that the approximation to efficiency can be improved by including the distribution of buyer's private valuations. If the median of the buyer is at least the median of the seller than posting a price between the two medians provides the better approximation. A second mechanism proposed [15] calculates "a weighted median based only on the distribution of the buyer and offers this price" [15]. The final mechanism introduced is the *Random-Quantile mechanism* that first chooses a number $x \in [\frac{1}{e}, 1]$ according to the price distribution $G$ with CDF $G(x) = \ln e \times x$ and then sets the price using the quantile function of the seller $q(\cdot)$ to be $q(x)$.